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BERTRAND'S POSTULATE AND ITS REFINEMENTS

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Abstract: Prime numbers are a key concept in Number Theory. The Bertrand’s postulate is one of the most extensively used theorems that guarantee the presence of a prime number inside a given interval. This paper provides results that are comparable to Bertrand's postulate, but with a smaller interval for a primes to exist. The paper covers the history of Bertrand’s postulate, proofs, importance, various applications, Ramanujan’s contributions on Bertrand’s postulate, some of its recent refinements and its future aspects.

Keywords: *Bertrand’s postulate, prime number, distribution of primes*.

2010 Mathematics Subject Classification: 11A41,11N05

1. Introduction

Number Theory is the queen of Mathematics and prime numbers are the most important component of it. Many real-world applications, such as cryptography and quantum programming, rely on prime numbers, and the most secure crypto-systems rely on them. They are also directly related to a variety of challenging mathematics problems, including the well-known Riemann hypothesis. More specifically, the Riemann hypothesis is intimately linked to prime number distribution. In many nations around the world, however, students are only introduced to primes and their distribution towards the very end of their studies. Students should be taught prime numbers, with a focus on their distribution, as early as their last two years of high school. The so-called Bertrand's postulate is a famous example of prime distribution. In this paper, our goal is to simplify Erdos’s proof of Bertrand’s postulate and offer a method for teaching high-school students about the distribution of primes using Bertrand’s postulate as a case study. Moreover, as the discussion of all the results that are stronger than Bertrand's postulate is beyond the scope of this article, we include a few of the most general refinements of Bertrand's postulate here.

2. Background

In 1845, Bertrand conjectured that there is a prime p satisfying $x < p < 2x$ for any real number $x > 1$and he verified for all $x < 3 \times 10^6$. It is the same thing if we restrict it to natural numbers $n > 1$ or to primes. (Given $x \geq 2$, let q be the largest prime less than or equal to x and let p be the prime satisfying $q < p < 2q$. Then $p > x$ by maximality of q and $p < 2q \leq 2x$. For $1 < x < 2$, take $p = 2$). Many proofs of Bertrand's postulate can be found in the literature. But Chebyshev proved it for the first time in 1852. He gave an analytical proof of this, and his proof is often presented in introductory courses after deriving some standard tools of analytic number theory. An excellent historical account can be found in [**9**]. Then in March 1931, the eighteen year old Paul Erdos found an elegant elementary proof of Bertrand's postulate: the following year in his first publication demonstrated the postulate/theorem using elementary properties and approximations of binomial coefficients and the Chebyshev functions:

$$\vartheta(x) = \sum_{p \leq x} \log p \text{ and } \psi(x) = \sum_{p^k < x} \log p$$

The proof by Erdos (which was later reproduced in Hardy and Wright's classic monograph [4], given in the celebrated "Proofs from The Book" [**1**] is calculus-free. He demonstrated that if there is no prime between n and 2n then we must have $2^{\frac{2n}{3}} < (2n)^{1+\sqrt{2n}}$ and proved that this cannot hold for $n \geq 4000$. Infact, this inequality holds for $n = 467$ but not for $n \geq 468$. The primes 2, 3, 5, 7, 13, 23, 43, 83, 163, 317, 631, 1259, 2509 and 4001 can be used to test Bertrand's postulate for $n \leq 3999$, with each prime being less than twice the previous one. The proof of Bertrand's postulate in Niven, Zuckerman and Montgomery's widely used Number Theory Textbook involves calculus and requires direct check for $n \leq 1599$. In 1919, Ramanujan [**10**] gave a short and elegant proof of Bertrand's postulate, which uses Stirling's formula and simple properties of the Γ–function (P. Ribenboim [12, p.188]). In the process he demonstrated the existence of a certain sequence of prime numbers (Theorem 2), now known as Ramanujan primes. Remember that $\pi(x)$ is the prime counting function i.e. $\pi(x)$ is the number of primes less than or equal to x. In August 2013, Jaban Meher and M.Ram Murty gave a slightly different proof of Bertrand's Postulate. He said (see [7]) "We are unable to find a calculus-free derivation of Stirling's formula but we eliminate the use of Stirling's formula from his (Ramanujan) proof. The revised proof now is so elegant that it qualifies to be included in "Proofs from The Book" [1]. We hope that our presentation and arrangement makes Ramanujan's proof more widely known and accessible to a larger community". Verma [17] gave a proof in the year 2018 by adding a few basic observations to Erdos's proof and using a little calculus that only involves a direct check for $n \leq 5$ and knowing that 2, 3, 5 and 7 are primes while 4, 6 and 8 are not, and no knowledge of primality of any other number. He also provided a proof without the use of calculus, requiring only a direct check for $n \leq 12$ and the knowledge that 2, 3, 5, 7 and 13 are primes but 4, 6, and 8 are not, as well as no prior knowledge of primality of any other number.

3. Preliminaries

The set of positive integers N is used throughout the study. The notion of mathematical induction and a general understanding of several mathematical constants, such as the factorial and the binomial coefficient, are all that is required as background. For the sake of completeness, the factorial of a nonnegative integer is given by $0! = 1$ and $n! = 1.2 \ldots (n-1).n$ for $n \in N$ and the binomial coefficient is given by

$$\binom{m}{n} = \frac{m!}{n!\,(m-n)!} \qquad (m \geq n)$$

It's common knowledge that if $m \geq n$, then the binomial coefficients $\binom{m}{n}$ is a positive integer. This is best explained with some concrete examples and a remark about combinatorial interpretations of this constant. The fundamental theorem of arithmetic states that every positive integer n can be written in a unique way as product of prime powers, $n = p_1^{\beta(p_1)} \ldots \ldots p_k^{\beta(p_k)}$, it is also known as the prime decomposition of n. For instance, $72 = 2^3 3^3$ where $p_1 = 2, \beta(2) = 3, p_2 = 3, \beta(3) = 2$. For simplicity of notation, the product of primes less or equal than x will be denoted $\prod_{p \leq x} p$. The following are the required results, which may usually be taught to students with proofs.

Lemma 1: For all $n \in N$, we have $\binom{2n}{n} > \frac{2^{2n}}{2n}$

Proof: We have $\binom{2n}{n} = \frac{(2n)!}{n!n!} = \frac{1.2 \ldots\ldots (2n-1)2n}{1.2 \ldots . n.n!} = 2^n \frac{3.5 \ldots\ldots (2n-1)}{n!}$

$> 2^n \frac{2.4 \ldots\ldots (2n-2)}{1.2 \ldots\ldots n} = 2^n \frac{2^{n-1}}{n} = \frac{2^{2n-1}}{n} = \frac{2^{2n}}{2n}$. This completes the proof.

Lemma 2: For all positive real x, we have

$$\prod_{p \leq x} p \leq 4^x$$

Proof: It is enough to establish the inequality for any positive integer n. We proceed by induction on n. The inequality is clear for $n = 1$ and $n = 2$. Let $n \geq 3$ and suppose that the result holds for all positive integer less than n. If n is even, then clearly

$$\prod_{p \leq n} p = \prod_{p \leq n-1} p \leq 4^{n-1} < 4^n$$

Let $n = 2k + 1$ be odd. Note that by induction

$$\prod_{p \leq k+1} p \leq 4^{k+1} \tag{1}$$

Moreover, as

$$\binom{2k+1}{k} = \frac{(k+1).(k+2)\ldots..2k(2k+1)}{1.2\ldots..k}$$

we immediately see that

$$\prod_{k+2<p\leq 2k+1} p \leq \binom{2k+1}{k}$$

(2)

and so it suffices an upper bound for

$$\binom{2k+1}{k}$$

We have

$$\binom{2k+1}{k} = \frac{1.2\ldots.2k(2k+1)}{1.2\ldots.k.(k+1)!} = 2^k \frac{3.5\ldots..(2k+1)}{1.2\ldots.k.(k+1)}$$

$$< 2^k \frac{4.6\ldots..(2k+2)}{1.2\ldots.k.(k+1)} = 2^k 2^k = 4^k$$

(3)

Now combine (1), (2), and (3) to obtain

$$\prod_{p\leq 2k+1} p = \prod_{p\leq k+1} p \prod_{k+2\leq p\leq 2k+1} p < 4^{k+1}.4^k = 4^{2k+1}$$

Lemma 3: For all positive real $x \geq 11$, we have $2^x \geq x^3$

Proof: Here again we use mathematical induction. For $n = 11$, we have $2^{11} = 2048 > 11^3 = 1331$. Now suppose the statement holds for n. Note first that $n^3 \geq 3n^2 + 3n + 1$ for all $n \geq 11$. Then we get $2^{n+1} = 2.2^n \geq 2n^3 = n^3 + n^3 \geq n^3 + 3n^2 + 3n + 1 = (n+1)^3$.

4. Bertrand's Postulate and its Proof

Theorem 1: For every positive integer n, there is a prime between n and $2n$.

Proof: We will be done if we show that the product of primes between

$$\prod_{n+1\leq p\leq 2n} p$$

primes between $n + 1$ and $2n$ is at least 1. From

$$\binom{2n}{n} = \frac{(n+1).(n+2)\ldots.2n}{1.2\ldots.n} \quad (4)$$

we deduce that the product of primes between $n+1$ and $2n$, if there are any, divides the binomial coefficient $\binom{2n}{n}$. Besides, noting that from (4) any prime divisor of $\binom{2n}{n}$ is less than $2n$ and using the fundamental theorem of arithmetic, we derive that $\binom{2n}{n} = T_1T_2T_3$ where

$$T_1 = \prod_{p<\sqrt{2n}} p^{\beta(p)}, \qquad T_2 = \prod_{\sqrt{2n}\leq p\leq n} p^{\beta(p)}, \qquad T_3 = \prod_{n<p\leq\sqrt{2n}} p,$$

Letting for a positive number x, $\pi\,(x)$ count the number of prime numbers less than x, we see that

$$T_1 < (2n)^{\pi(\sqrt{2n})}. \tag{5}$$

As to T_2, observe that if $\frac{2n}{3} < p \leq n$, then p is a factor in the denominator of (4) but $2p$ is not and $2p$ is a factor in the numerator but $3p$ is not. As for $p\ >\ 2$ we have p^2/p we see that p cancels and therefore $T_1 = T_2 = \prod_{\sqrt{2np}\leq 2n/3} p^{\beta(p)}$ and so by virtue of Lemma 2,

$$T_2 \leq 4^{2n/3} \tag{6}$$

Summarasing, we get

$$\frac{2^{2n}}{2n} \leq \binom{2n}{n} = T_1T_2T_3 < (2n)^{\pi\,(\sqrt{2n})}\,4^{2n/3}T_3$$

implying that

$$T_3 > \frac{2^{2n/3}}{(2n)^{\pi(\sqrt{2n})+1}} \tag{7}$$

Suppose that $\sqrt{2n} > 15$ or equivalently $n\ \geq 113$. Clearly, $\pi(\sqrt{2n})$ is less than the number odd positive integers which are less than $\pi(\sqrt{2n})$. This combined with the fact that 1,9 and 15 are not primes we derive that

$$\pi(\sqrt{2n}) + 1 \leq \left(\frac{\sqrt{2n}+1}{2} - 2\right) + 1 \leq \frac{\sqrt{2n}}{2}$$

which combined with (7) yields that for $n\ \geq 113$,

$$T_3 > \frac{2^{2n/3}}{\left(\sqrt{2n}\right)^{\sqrt{2n}}} = \left(\frac{2^{\sqrt{2n}}}{(\sqrt{2n})^3}\right)^{\sqrt{2n}/3}$$

This combined with the fact $2^{\sqrt{2n}} > \left(\sqrt{2n}\right)^3$ by Lemma 3, we find that for $n\ \geq 113$,

$$\prod_{n+1\leq p\leq 2n} p = T_3 > 1$$

Finally, by inspection we can show that for any $n\ <\ 113$ there is a prime between n and $2n$.

5. Some Refinements of Bertrand's Postulate

The existence of prime numbers has been extensively investigated over the last two centuries, with some discoveries that are stronger than Bertrand's postulate. While it is beyond the scope of this article to evaluate all of these findings, we have listed a few of the most general refinements of Bertrand's postulate below. We propose [14] and [18] for further reading.

As mentioned earlier, Ramanujan in his two page paper [10,11], published one year before his death in 1920 at the age of 32, the Indian mathematical genius Srinivasa Ramanujan proved Bertrand's postulate and analyzed the phenomenon at the end of his paper, where he proves the following extension of Bertrand's Postulate.

Theorem 2: (Ramanujan) Let $\pi(x)$ denote the number of primes not exceeding$x$. Then $\pi(x) - \pi\left(\frac{x}{2}\right) \geq 1,2,3,4,5, \dots \dots \dots, if\ x \geq 2,11,17,29,41, \dots \dots . respectively.$

Ramanujan uses the following inequality to prove the theorem.

$$\pi(x) - \pi\left(\frac{1}{2}\ x\right) > \frac{1}{\log x}\left(\frac{1}{6}\ x - 3\sqrt{x}\right) \text{ if } x > 300 \dots\dots \quad (1)$$

As we know that the nth Ramanujan prime (for $n \geq 1$) is the smallest positive integer R_n with the property that if $x \geq R_n$, then $\pi(x) - \pi\left(\frac{x}{2}\right) \geq n$. Obviously R_n is a prime, because by the minimality condition the functions $\pi(x) - \pi\left(\frac{x}{2}\right)$ and, therefore, $\pi(x)$ must increase at $x = R_n$. Since they can increase by at most 1, the equlity $\pi(R_n) - \pi\left(\frac{1}{2} R_n\right) = n$ holds. As an example, if $n = 1,2,3,4,5 \dots$ then the nth Ramanujan Prime $R_n = 2,11,17,29,41,47, \dots$

Note that Bertrand's Postulate is $R_2 = 2$. To compute R_2, set the quantity on the right side of inequality (1) equal to 1 and solve, obtaining x=392.39… Since the quantity is an increasing function of x in the range $x > 300$, and the left side of (1) is an integer and can change only at integers, it follows that $\pi(x) - \pi\left(\frac{x}{2}\right) \geq 2$ if $x > 392$ This gives the bound $R_2 \leq 392$. Counting primes, we find that $\pi(x) - \pi\left(\frac{x}{2}\right) \geq 2$ for $n = 391,390,389, \dots 11$, but $\pi(10) - \pi(5) = 1$. Thus $R_2 = 11$. In the same way, one gets $(R_3, R_4, R_5) = (17,29,41)$

Jonathan Sondow [15] proved that the nth Ramanujan Prime R_n lies between the $2n$ th and $4n$th prime for all $n \geq 2$. So he derived the upper bound which is smaller than that derived from (1) and gave a faster method of computing R_n. He also showed that $R_n \sim p_{2n}$ as $n \to \infty$ and that for every $\in > 0$, there exists $N_0(\in)$ such that $R_n < (2+\in) n \ln n$ for $n \geq N_0(\in)$. Shanta Laishram [5] improved Sondow's result by showing that the nth Ramanujan prime does not exceed the 3nth prime. The theorem states that the asymptotic formula $R_n \sim p_{2n}$ as $n \to \infty$ implies $R_n < p_{3n}$, for n large. It also holds for $n \leq 1000$.

In a similar fashion, in 1958 Polish mathematician W. Sierpinski postulated that for all $n > 1$ and $k \leq n$ there exists at least one prime number in the closed interval $[kn, (k+1)\,n]$. It is obvious that the statement holds for $k = 1$ as a direct consequence of Bertrand's postulate.

In 2006, Bachraoui [2] gave a proof for $k = 2$ i.e. He showed that for $n > 1$ there is a prime number between $2n$ and $3n$ (Theorem 3, below). This similar result to Bertrand's postulate provided a smaller interval for a prime number to exist. As an example, Bertrand's postulate guarantees $p \in (10,20)$, whereas Bachraoui guarantees $p \in (10,15)$. Moreover, Bachraoui also questioned if there was a prime number between kn and $(k+1)n$ for all $n \geq k \geq 2$. A simple corollary of this result is that for all $n \geq 1$ the interval $\left(n, \frac{3(n+1)}{2}\right)$ contains at least one prime number (Theorem 4 below). The formal proof of this refinement of Bertrand's postulate can be found in [16]. The case $k = 3$ was proved by Loo [6] in 2011 and it leads to a further refinement that guarantees the existence of a prime number in the interval $\left(n, \frac{4(n+2)}{3}\right)$ for all $n \geq 3$. Thus Andy Loo [6] shortened these intervals to $p \in (3n, 4n)$ for $n > 1$. Andy Loo went on to prove that as n approaches infinity the number of prime numbers between $3n$ and $4n$ also tends to infinity - a result which is implied by the prime number theorem. From the theoretical perspective, both of the refinements above are a consequence of a refinement proved by Nagura [8] in 1952. Namely, he proved that for $n \geq 25$ the interval $\left(n, \frac{6n}{5}\right)$ contains at least one prime number [8]. However, his proof relies on more advanced results and concepts from Number Theory and Calculus. Kyle D. Balliet [3] showed that there is always a prime number in the interval $(4n, 5n)$ for any positive integer $n > 2$. He also proved that there are at least seven prime numbers between n and $5n$ for all $n > 5$. In the year 2013, Shevelev, et al. [13] proved that the list of integers k for which every interval $(kn, (k+1)n), n > 1$, contains a prime includes $k = 1, 2, 3, 5, 9, 14$ and no others, at least for $k \leq 100,000,000$. Using the prime number theorem it can also be proved that for any $\in > 0$ there exists n_0 such that for all $n > n_0$ the interval $(n, (1+\in)\,n)$ contains at least one prime number. Note that this generalized statement does not give a precise value of n_0 and might be unsuitable for finding all solutions of a given equation or solving some similar types of problems.

Theorem 3. For any positive integer $n > 1$, there is a prime number between $2n$ and $3n$.

Proof: (For the proof, see [2] or [19].) The demonstration in [2] was typical of many theorems in number theory and was based on multiple inequalities valid for large values of n which can be calculated effectively. For the rest of the values of n, there are many basic improvisations, some perhaps difficult to follow.

Theorem 4. For $n \geq 1$, there is a prime number p such that $n < p < 3(n+1)/2$.

(Since $3(n+1)/2 < 2n$ for $n > 3$, this is a refinement of the Bertrand's postulate.)

Proof: The case $n = 1$ follows from $1 < p = 2 < 3$. The case $n = 2$ follows from $2 < p = 3 < 9/2$. For n even, say $n = 2k$, by Theorem 3, we have a prime p such that

$n = 2k < p < 3k < \frac{3(2k+1)}{2} = \frac{3(n+1)}{2}$. Similarly, for n odd, say $n = 2k + 1$, we have a prime p such that $n = 2k + 1 < 2k + 2 = 2(k + 1) < p < 3(k + 1) = 3(n + 1)/2$. This completes the proof.

Theorem 5 (Kyle D. Balliet): For any positive integer n $>$ 2 there is a prime number between 4n and 5n.

Proof: See [3]

On the basis of the previous theorem, we can prove that for all positive integers $n > 2$, there is always a prime number between n and $(5n + 15)/4$ for all positive integers $n > 2$ as in the following theorem. i.e. the theorems 6,7,8,9 and 10 are the consequences of the theorem 5 (due to Kyle D. Balliet, 2015 [3]).

Theorem 6. For any positive integer n $>$ 2 there exists a prime number p satisfying n $<$ p $< \frac{5(n+3)}{4}$

Proof. When $n = 3$, we obtain $3 < 5 < 7 < \frac{15}{2}$

Let $n \geq 4$. By the division algorithm $4 \mid (n + r)$ for some $r \in \{0, 1, 2, 3\}$ and by Theorem 5 there exists a prime number p such that $p \in \left(n + r, \frac{5(n+r)}{4}\right)$

Since $\left(n + r, \frac{5(n+r)}{4}\right)$ is contained in $\left(n, \frac{5(n+3)}{4}\right)$ for all $0 \leq r \leq 3$ and $n > 2$, $p \in \left(n, \frac{5(n+3)}{4}\right)$ as desired.

Theorem 7. For any positive integer n $>$ 2 there are at least four prime numbers between n and 5n.

Proof. The cases when $n = 3, 4, \ldots, 14$ may be verified directly. Now let $n \geq 15$ and by Theorem 6 we know there exists prime numbers p_1, p_2 and p_3 such that $< p_1 < \frac{5n+15}{4}, 2n < p_2 < \frac{10n+15}{4}$, $3n < p_3 < \frac{15n+15}{4}$ and by Theorem 5 there exist a prime number p_4 such that $4n < p_4 < 5n$. Hence

$$n < p_1 < \frac{5n + 15}{4} < 2n < p_2 < \frac{10n + 15}{4} < 3n < p_3 < \frac{15n + 15}{4} \leq 4n < p_4 < 5n$$

We improve on the number of prime numbers between n and $5n$ in the next theorem to show that for $n > 5$, there are at least seven prime numbers between n and $5n$.

Theorem 8. For all n $>$ 5 there are at least seven prime numbers between n and 5n.

Proof. The cases when $n = 6, 7, \ldots, 244$ may be verified directly. Now let $f(n) = \frac{5n+15}{4}$ for $n \geq 245$ and let $f^m(n) = f\left(f^{m-1}(n)\right)$. By Theorem 6 there exists a prime number between n and $f(n)$. Furthermore, there exists a prime number between $f^{m-1}(n)$ and $f^m(n)$ for all $m \in N \setminus \{1\}$. In general,

$$f^m(n) = \frac{1}{4^m}\left(5^m n + 3\sum_{k=0}^{m-1} 5^{m-k}4^k\right)$$

Consider

$$f^m(n) = \frac{1}{4^m}\left(5^m n + 3\sum_{k=0}^{m-1} 5^{m-k}4^k\right) \le 5n$$

Solving for n, we obtain

$$n \ge \frac{3}{5.4^m - 5^m}\sum_{k=0}^{m-1} 5^{m-k}4^k$$

However, $5.4^m - 5^m$ is positive only for $m \le 7$. So let $m = 7$, then for

$$n \ge 245 > \frac{3}{5.4^7 - 5^7}\sum_{k=0}^{6} 5^{7-k}4^k$$

there are at least seven prime numbers between n and $5n$ and hence the proof is complete.

Theorem 9. For $\mathrm{n} > 2$ the number of prime numbers in the interval $(4\mathrm{n}, 5\mathrm{n})$ is at least

$$\log_{5n}\left[\frac{0.054886}{2^{\frac{n}{2}} n^{\frac{3}{2}}}\left(\frac{3125}{256}\right)^{\frac{n}{6}}(5n)^{-\frac{2.51012\sqrt{5n}}{\log(5n)}}\right]$$

Proof. In Theorem 5 we approximated the product of prime numbers between $4\mathrm{n}$ and $5\mathrm{n}$ from below by

$$\frac{0.054886}{2^{\frac{n}{2}} n^{\frac{3}{2}}}\left(\frac{3125}{256}\right)^{\frac{n}{6}}(5n)^{-\frac{2.51012\sqrt{5n}}{\log(5n)}}$$

Bounding each of the prime numbers between $4\mathrm{n}$ and $5\mathrm{n}$ from above by $5n$, we obtain:

$$\begin{aligned}
&\log_{5n}\left[\frac{0.054886}{2^{\frac{n}{2}} n^{\frac{3}{2}}}\left(\frac{3125}{256}\right)^{\frac{n}{6}}(5n)^{-\frac{2.51012\sqrt{5n}}{\log(5n)}}\right] \\
&\quad = \frac{\log 0.054886 - \frac{n}{2}\log 2 + \frac{n}{6}\log\frac{3125}{256}}{\log n + \log 5} + \frac{\frac{3}{2}\log 5 - 2.51012\sqrt{5n}}{\log n + \log 5} - \frac{3}{2} \\
&\quad > \frac{n\left(\frac{1}{6}\log\frac{3125}{256} - \frac{1}{2}\log 2 - \frac{2.51012\sqrt{5}}{\sqrt{n}}\right)}{2\log n} + \frac{\frac{3}{2}\log 5 + \log 0.054886}{\log n + \log 5} \\
&\quad - \frac{3}{2} > \frac{n}{\log n}\left(0.035214 - \frac{2.8063995}{\sqrt{n}}\right) - \frac{168033}{100000}
\end{aligned}$$

Now observe that

$$\lim_{n\to\infty}\frac{2.8063995}{\sqrt{n}}=0$$

Moreover

$$\lim_{n\to\infty}\frac{n}{\log n}=+\infty$$

By Theorem 9 we have the following theorem.

Theorem 10: As n tends to infinity, the number of prime numbers in the interval $[4n, 5n]$ goes to infinity. That is, for every positive integer m, there exists a positive integer L such that for all $n \geq L$, there are at least m prime numbers in the interval $[4n, 5n]$.

6. Conclusion

Though there are lots of results/refinements that are stronger than the Bertrand's Postulate, but one could expect that the following unsolved famous mathematical problem will dramatically improve the Bertrand's result.

Is there always a prime between n^2 and $(n+1)^2$?

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References

[1] Aigner, M. and Ziegler, G. (2009). Proofs from the Book, fourth edition. Springer-Verlag, New York.

[2] Bachraoni, M.El. (2006). Primes in the Interval [2n, 3n], Int. J. Contemp. Math. Sci. **1**(13) 617-621.

[3] Balliet, K.D. (2017). On the Prime Numbers in the Interval [4n, 5n], arxiv.org/abs/1511.04571.

[4] Hardy, G.H. and Wright E.M. (1938). An Introduction to the Theory of Numbers, Oxford University Press, Oxford.

[5] Laishram, S. (2010). On a conjecture on Ramanujan primes, Int. J. Number Theory, **6**(8) 1869-1873.

[6] Loo, A. (2011). On the Primes in the Interval [3n, 4n], Int. J. Contemp. Math. Sci. **6**(38), 617-621.

[7] Meher, J. and Ram Murty, M. (2013). Ramanujan's Proof of Bertrand's postulate, The American Mathematical Monthly **120**(7) 650-653.

[8] Nagura, J. (1952), On the interval containing at least one prime number, Proceedings of the Japan Academy, Series A, **28**, 177-181.

[9] Narkiewicz, W. (2000), The Development of Prime Number Theory, Springer-Verlag, New York.

[10] Ramanujan, S. (1919) A proof of Bertrand's postulate, J. Indian Math. Soc. **11,** 181-182.

[11] Ramanujan, S. (2000), Collected Papers, G.H. Hardy, et. al. (Eds), American Mathematical Society, Providence, Rhode Island.

[12] Ribenboim, P. (1989), The Book of Prime Number Records, 2nd ed., Springer-Verlag, New York.

[13] Shevelev, V., Greathouse IV, C.R. and Moses, P.J.C. (2013), On the intervals $(kn,(k+1)n)$ containing a prime for all n, Journal of Integer Sequences **16**, Article 13.7.3

[14] Sierpinski, W. (1988), Elementary Theory of Numbers, North-Holland Mathematical Library; **31**, ISBN 0444-866620.

[15] Sondow, J.(2009), Ramanujan primes and Bertrand's postulate, Amer. Math. Monthly **116**, 630-635; also available at http://arxiv.org/abs/0907.5232.

[16] Stanciu, N. (2009-2010), A Refinement of Bertrand's Postulate, Mathematical Excalibur **14**(4).

[17] Verma, M. (2018), Proof of Bertrand's Postulate for $n \geq 6$, arxiv.org/abs/1401.4368

[18] https://en.wikipedia.org/wiki/Bertrand\%27s_postulate (accessed on May 31, 2016).

[19] http://www.m-hikari.com/ijcms.html.