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HARMONIOUS COLORING ON PRODUCT OF GRAPHS

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Abstract: Harmonious coloring is a proper vertex coloring such that no two edges have same color pair. The minimum number of coloring of a graph is called a harmonious Chromatic number which denoted by $\chi_h(G)$. In this paper we find that the harmonious chromatic number for rooted product of graphs and established the relation between harmonious chromatic number of Cartesian product of graphs and rooted product of graphs.

Keywords: Coloring, Harmonious coloring, Cartesian product, Rooted product.

Mathematics Subject Classification: 05C62, 68R10

1. Introduction

In mathematical sciences, graph coloring is very easy to formal and visualize, but they have more characteristics that are very problematic to answer [7]. This study traces back to the origin of the four-color problem of whether it is possible to color the regions of every map with four colors such that every two regions having a common boundary are assigned different colors. Well ahead, it was understood that this is a problem in graph theory, whether it is at all times possible to color the regions of each planar graph so that any two adjacent regions are colored as differently [10]. It took more than 160 years and communal efforts to prove the simple-sounding proposition that four colors are sufficient to color the vertices of a planar graph properly [2]. Early research in coloring was focused commonly on many theoretical aspects, particularly the statements relating to the chromatic number for particular topologies such as planar graphs, line graphs, critical graphs, triangle-free graphs, and perfect graphs. Numerous of the real-world operational research problems can be embark upon using graph coloring methods, and these include unrelated problem areas of producing sports schedules, solving Sudoku puzzles, checking for short circuits on printed circuit boards, assigning taxis to customer requests, timetabling lectures at a university, finding worthy seating ideas for guests at a wedding, and assigning computer-programming variables to computer registers [7]. The harmonious coloring of a graph is to find the least possible number of colors needed to color the vertices of a graph such that the color pairs of end vertices of every edge are distinct. Hopcroft and Krishnamoorthy first well-known the abstraction of the harmonious coloring and demonstrated the NP-completeness for overall graphs[5]. Later[8], one more condition was added that adjacent vertices has received different colors, and the least possible number is called the harmonious chromatic number. This number has been calculated for graphs such as paths, complete binary trees, two- and three-dimensional grids and graphs with a diameter maximum two [8]. Therefore, there be present to finding the exact bound for the graphs such as the collection of non-trivial disjoint of paths graph, cycle graph, complete graphs, trees, triangular snake graphs, double triangular snake graphs, and diamond snake graphs [1,4,9,11]. Away from each other, Lee and Mitchem [6] provided an upper bound for harmonious chromatic number of a graph.

In 1970 Frucht and Harary derivated, the corona product of two graphs G_1 and G_2 is the graph $G_1 \times G_2$ constructed by one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2 .

For graphs G and H, the Cartesian product of G and H is the graph G□H with vertex set V(G□H) = {(x,y) : x∈V(G), y∈V(H)}, and edge set E(G□H) = {(x,u)(y,v) : x =y with uv∈E(H), or xy∈E(G) with u=v}

The concept of rooted product graph was introduced in 1978 by Godsil and McKay[3]. Given a graph G of order n and a graph H, the rooted product graph G ◦ H is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the i^{th} vertex of G with the root vertex v in the i^{th} copy of H for every i ∈{1, 2, . . . , n}.

2. Some Results on Harmonious Coloring

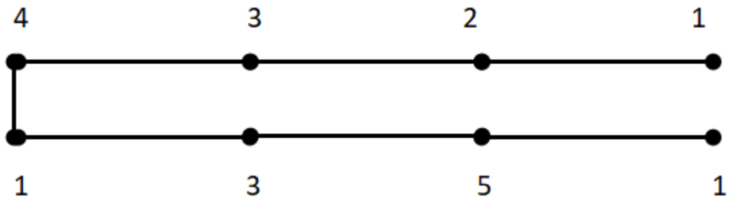
Theorem 2.1[3]**:** Let G be any graph, if $\chi(G)$ is chromatic number of G, then $\chi(G) < \chi_h(G)$

Theorem 2.2[3]**:** If $G_1 \subseteq G$, then $\chi_h(G) \geq \chi_h(G_1)$

Theorem 2.3[3]**:** For any complete graph K_n, $\chi_h[B(K_n,K_n)]=2n-1$, $n \geq 2$. where $B(K_n,K_n)$ Fulfills the following conditions (i)The number of vertices in $B(K_n,K_n)$ is 2n. (ii) The number of edges in $B(K_n,K_n)$ is n^2-n+1. (iii) The maximum degree in $B(K_n,K_n)$ is n. (iv) The minimum degree in $B(K_n,K_n)$ is n-1.

3. Harmonious Coloring on Rooted Product of Graph

Observation 3.1: Let P_2 and $P_m (m = 2,3,4)$ be any two paths. If G is rooted product of P_2 and P_m, then $\chi_h(G) = m+1$. The following figure clearly shows that the harmonious chromatic number of $\chi_h(P_2 \circ P_2)$, $\chi_h(P_2 \circ P_3)$ and $\chi_h(P_2 \circ P_4)$



$\chi_h\left(P_2 \circ P_4\right) = 5$

Lemma 3.2: Let P_2 and $P_m (m = 5, 6, 7)$ be two paths. If G is a rooted product of P_2 and P_m, then $\chi_h(G) = m$

Proof: Let us label the vertices of G as $v_{11}, v_{12}, \dots, v_{1m}, v_{21}, v_{22}, \dots, v_{2m}$. Now assign a color {1,2,…,m} to $v_{1m}, v_{1(m-1)}, \dots, v_{11}$ respectively. If m is even, without loss of generality assign a color 1,3,…,m-1,2,4,…,m-2 to $v_{21}, v_{22}, \dots, v_{2m}$ respectively or if m is odd, assign a color 1,3,…,m,2,4,…,m-2 to $v_{21}, v_{22}, \dots, v_{2m}$ respectively, this gives an exact harmonic chromatic number for G.

Theorem 3.3: Let P_2 and $P_m (m > 9)$ be any two paths. If G is $P_2 \circ P_m$, then $\left\lceil \frac{m}{2} \right\rceil \le \chi_h(G) \le m - 1$

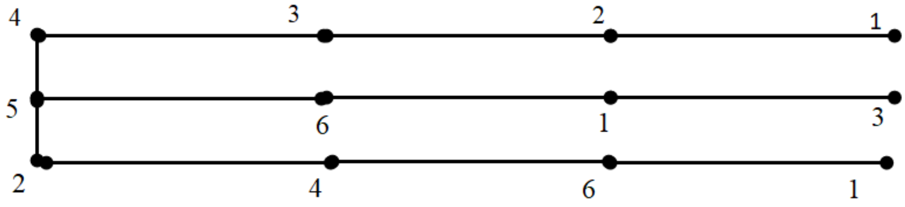
Proof: Let us consider the vertices of G as $v_{11}, v_{12}, \dots, v_{1m}, v_{21}, v_{22}, \dots, v_{2m}$. Thus $|V(G)| = 2m$

Suppose $h(G) : V(G) \to \left\{1, 2, \dots, \frac{m}{2}\right\}$ which is assigned the following way, let us assign the color for $v_{1m} = 1, v_{1(m-1)} = 2, \dots, v_{1(m/2)} = m/2,\ v_{1\left(\frac{m}{2}-1\right)} = \frac{m}{2} - 2, v_{1\left(\frac{m}{2}-2\right)} = \frac{m}{2} - 4, \dots$

Continuing the same process we can assigned a color on $v_{11}, v_{12}, \dots, v_{1m}$ from $V(G)$. At this situation assigning color to $v_{21}, v_{22}, \dots, v_{2m}$ is questionable, if we follow the above process we assign a color uniquely for $v_{21}, v_{22}, \dots, v_{2\left(\frac{m}{2}\right)}$ and then the remaining vertices are not possibly to assign a color form $\left\{1,2,...,\frac{m}{2}\right\}$ which is contradiction to $h(G):V(G)\to\left\{1,2,...,\frac{m}{2}\right\}$. Now we include at least one new color to in the codomain, this fails to assigning unique color of $G,$ once again add a new color on any vertex one by one up to $m-1$. Therefore $\chi_h(G)$ lies between $\left\lceil\frac{m}{2}\right\rceil$ and $m-1$.

Theorem 3.4: Let P_3 and $P_m(m=2,3,4)$ be any paths. If G is $P_3\circ P_m$, then $\chi_h(G)=m+2$

Proof: Let us label the vertices of G as $v_{11},v_{12},...,v_{1m},v_{21},v_{22},...,v_{2m},v_{31},v_{32},...,v_{3m}$. Now define $h(G):V(G)\to\{1,2,...,m,m+1,m+2\}$ choose the first (m+2) vertices have coloring as follows $v_{1m}=1,v_{1(m-1)}=2,...,v_{11}=m,v_{21}=m+1,v_{22}=m+2$ left out beginning vertex has start with color 1 and add difference 2 to next vertex and so on which gives $v_{23}=1,v_{24}=3,v_{25}=4,...$ continuing the same process we can get required result. This is clear from below example



$\chi_h(P_3\circ P_4)=6$

Theorem 3.5: Let P_3 and $P_m(m\neq 2,3,4)$ be any two paths. If G is $P_3\circ P_m$, then $\left\lceil\frac{m}{2}\right\rceil\leq\chi_h(G)\leq m$

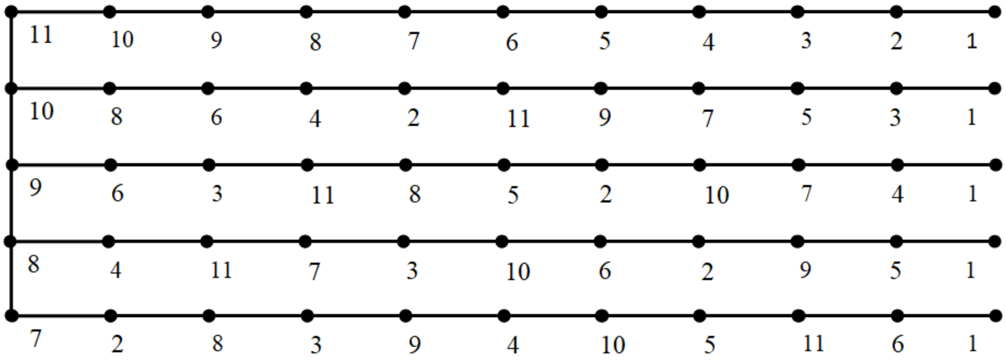
Proof: Let us consider the coloring as χ:{i,i+1,…j}→V(G) where i ≥ $\frac{n}{2}$ and j<m. Let us label the vertices of G as 11,12,…,1m,21,22,…,2m,31,32,…,3m. Now assigning color to the vertex as follows.

Without loss of generality consider the vertex 1m has colored i, 1(m-1) has colored i+1,…then we can reach vertex 12 has color j. Now once again we start a color i to the vertex 1,since 11 is not adjacent to 1m and 21 has colored i+2, 22 has colored i+4 etc., by theorem 3.3 we can assign a color to 11,12,…,1m,21,22,…,2m. Now once again start a color 1 to the vertex 31, since 21 & 31 are different colored now the vertex 32 assign a color i+3, and continuing the similar manner we can get required result.

Corollary 3.6: Let P_n and P_m be any paths. If G is $P_n \circ P_m$, then $(i)\ \chi_h(G) \leq m$ where $n \leq \frac{m}{2}$ $(i)\ m \leq \chi_h(G) \leq \left\lceil \frac{n}{2} \right\rceil m$ where $n > \frac{m}{2}$

Proof: Let us label the vertices of G as v_{ij} for $1 \leq i \leq n \,\&\, 1 \leq j \leq m.$

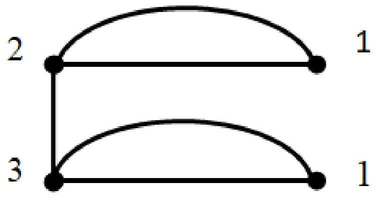
If $n \leq \frac{m}{2}$, the harmonious coloring will be in the following manner, we assign a color 1 to the vertex v_{1m} and continue the coloring with the distinct color on ascending order to consecutive vertices to reach first row starting vertex, that is m^th^ color assigned to v_{11}. The second row once again start a color 1 to v_{2m} and then second vertex onwards every colors has difference 2, if the color number is greater than m, which is considered as the reminder of m to that place, continuing same process we can reach second row first vertex has colored as m-1. As per above process the third row last vertex v_{3m} colored as 1 and then, the consecutive vertices are coloring with difference 3 finally we arrived a color m-2 for v_{31} Continuing the above process we can get required results, vide below figure.



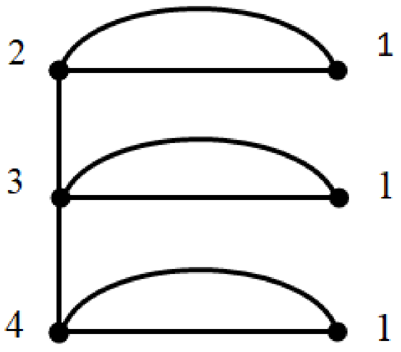
$\chi_h(P_5 \circ P_{11}) = 11$

3.1 Harmonious Coloring on Rooted Product of Path and Cycle

Observation 3.7: Let P_n be a path and C_2 be cycle. If G is rooted product of P_n and C_2, then $\chi_h(G) = n+1.$ The following figure is clearly shows that the harmonious chromatic number.

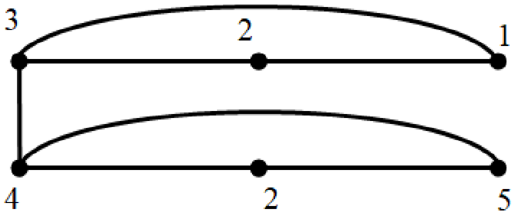


$\chi_h\left(P_2 \circ C_2\right) = 3$

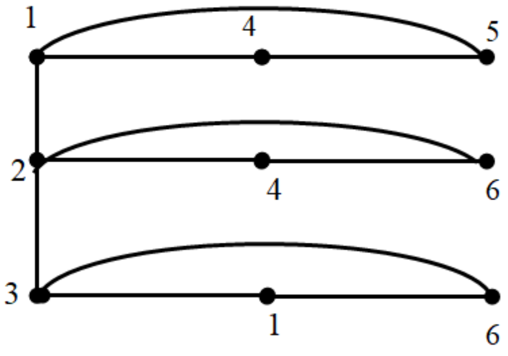


$\chi_h\left(P_3 \circ C_2\right) = 4$

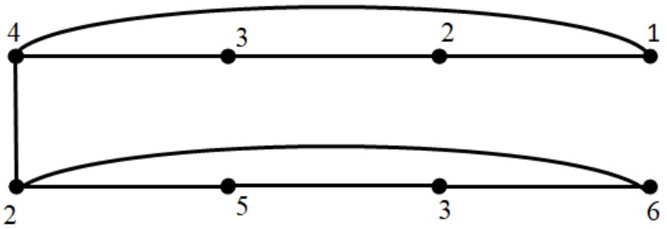
Observation 3.8: Let P_n being any path and $C_m (m=3,4)$ be any cycle. If G is $P_3 \circ P_m$, then $\chi_h(G) = n+m.$



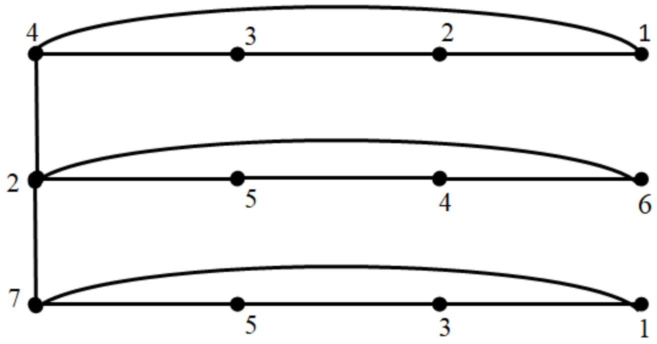
$\chi_h\left(P_2 \circ C_3\right) = 5$



$\chi_h\left(P_3 \circ C_3\right) = 6$



$\chi_h(P_2 \circ C_4) = 6$



$\chi_h(P_3 \circ C_4) = 7$

Theorem 3.9: Let P_n be any path and $C_m (m \geq 5)$ be any cycle. If G is $P_3 \circ P_m$, then $\chi_h(G) \geq n + m/2$.

Proof: By observation on $\chi_h(C_m) \geq \lceil m/2 \rceil$. Here each vertex of P_n incident with a cycle of length m. Therefore $\chi_h(G) \geq n + m/2$.

Theorem 3.10: Let G_n be any graph with order n and K_m be a complete graph, if G is $G_n \circ K_m$ then $\chi_h(G) \leq \sum_{k=m+1-n}^{m} k$ where $n \leq m$

Proof: Proof is by method of induction on n, if $n = 1$ then which is clear that $\chi_h(G) = m = \sum_{k=m+1-n}^{m} k$. Therefore, now let us assume that $n > 1$. let us consider,

$nK_m = \bigcup_{k=1}^{n} K_m^k$, *where* $K_m^k \cong K_m$ for every $k \in \{1,2,...,n\}$. which is clear that the coloring assigns m different colors to the vertices of K_m^1. Likewise we can assign again m different colors to the vertices K_m^2. On the other hand, if there are at least two colors of vertices of K_m^1 that also appear on the vertices of K_m^2, without loss of generality let us consider the color i and j then there is an edge from i to j in K_m^1 and K_m^2 is not possible. Therefore, we need at least $m+(m-1)$ colors are assign to the copies of K_m^1 and K_m^2

At this situation consider the m different colors assigned to the vertices of K_m^3. Suppose K_m^3 contains at least 3 colors which are appeared in K_m^1 & K_m^2. by the principle of pigeonhole at least two of these colors lies on the same component. That is pair of these 3 colors appears either on K_m^1 or K_m^2 if any 2 color out of 3 assigned to K_m^1 and which are adjacent, these are adjacent on K_m^3, which is a contradiction. Therefore we need at least $m+(m-1)+(m-2)$ colors in order to color the components K_m^1, K_m^2 and K_m^3. Continue the same process we can get, $\chi_h(G) \leq \sum_{k=m+1-n}^{m} k$ where $n \leq m$

Theorem 3.11: Let G be any graph with order n and H be $K_{1,m}$ graph, if $G_1 = G \circ H$ then $\chi_h(G_1) \leq nm$

Proof: We know that $\chi_h(H) \leq m$ since a vertex has degree m and m vertices has degree one. Let us consider the vertices $v_{1i}, v_{2i}, ..., v_{ni}$ are assigned to the color 2,3,…,m,m+1 respectively for $1 \leq i \leq m$, since each $v_{ij}, 1 \leq i \leq n$ & $1 \leq j \leq m$ has degree one, based on these coloring we can obtain a results quickly. And let us consider color 1 has assigned to vertex has degree m.

Case (i): If G is not a complete graph; clearly every vertex has degree less than n-1. Now we can assigning a color to G using the definition of harmonious coloring other than colors from {2, 3,…,m,m+1} but the color 1 should be in one of the harmonious coloring of a graph G. At this situation, from G_1 every isolated vertex has colored from 2,3,…,m+1 and then the remaining vertices are assigned a colors from other than above collection. Clearly this gives less than nm colors of G_1.

Case (ii): Suppose G is complete graph, clearly every vertex has assigned different colors so we need n colors, then G_1 has exactly equal to nm colors.

Theorem 3.12: Let G_i is and G_j being any two graphs, If G is $G_i \Box G_j$ and H is $G_i \circ G_j$ then $\chi_h(G) \geq \chi_h(H)$

Theorem 3.13: Let G_i is and G_j being any two graphs, If G is $G_i \times G_j$ and H is $G_i \circ G_j$ then $\chi_h(G) \geq \chi_h(H)$

Proof: If either G_i or G_j has at least one isolated vertex, Then degree of every vertex in G has greater than one, since by the definition of Cartesian product of two graphs, but not in H some of vertices remain isolated. Therefore we can get required results.

Theorem 3.15: Let G_i and G_j being any two graphs, G_i and G_j are harmonious coloring if and only if $G_i \circ G_j$ is harmonious colorable.

Proof: Proof is obvious from Previous theorem:

Theorem 3.16: If G_i and G_j be any two graphs, then $\chi_h\left(G_i \circ G_j\right) \leq \chi_h\left(G_i\right) + \chi_h\left(G_j\right)$

Proof: By contradiction, let us assume that $\chi_h\left(G_i \circ G_j\right) > \chi_h\left(G_i\right) + \chi_h\left(G_j\right)$ without loss generality Let us consider L be the harmonious coloring of G_i and G_j with is exactly equal to $\chi_h\left(G_i\right) + \chi_h\left(G_j\right)$ colors. Suppose ab is an edge of G_i or G_j and either a or b is an isolated vertex belongs to G_i, then the coloring L satisfies $V(G_i)$ and $V(G_j)$ which is also a harmonious coloring. But $G_i \circ G_j$ is contradicts to the definition since degree of a is greater than one.

4. Conclusion

In this Paper we find that the upper bounds for Harmonious coloring on rooted product of any two paths or any graph with complete graph and the Harmonious coloring of Cartesian product of graph and corona product of graphs always greater than or equal to Harmonious coloring of rooted product of graphs.

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References

[1] Aflaki, A. Akbari, S.; Edwards, K.J.; Eskandani, D.S.; Jamaali, M.; Ravanbod, H. (2012). On harmonious colouring of trees. Electron. J. Comb. **19**, 3-11.

[2] Appel, K. Haken, W. (1976). Every planar map is four colorable. Bull. Am. Math. Soc. **82**, 711-712.

[3] Francisco Antonio Muntaner-Batle J. Vernold Vivin M. Venkatachalam (2014). Harmonious Coloring on Corona Product of Complete Graphs. Natl. Acad. Sci. Lett. **37**,461-465.

[4] Godsil, C.D. McKay, B.D. (1978). A new graph product and its spectrum Bull. Austral. Math. Soc. **18**, 21-28.

[5] Georges, J.P. On the harmonious coloring of collections of graphs. (1995). J. Graph Theory, **20**, 241-254.

[6] Hopcroft, J. Krishnamoorthy, M.S. (1983). On the harmonious coloring of graphs. SIAM J. Algebr. Discrete Methods, **20**, 306-311.

[7] Lee, S. Mitchem, J. (1987). An upper bound for the harmonious chromatic number of a graph. J. Graph Theory, **11**, 565-567.

[8] Lewis, R.M.R. (2016). A Guide to Graph Coloring; Springer International Publishers: Cham, Switzerland.

[9] Miller, Z. Pritikin, D. (1991). The harmonious coloring number of a graph. Discrete Math, **93**, 211-228.

[10] Selvi, M.S.F.T. (2015). Harmonious coloring of central graphs of certain snake graphs. Appl. Math. Sci, **9**, 569-578.

[11] Zhang, P. (2015). Color-Induced Graph Colorings; Springer: London, UK.