

EULER TYPE INTEGRALS ASSOCIATED WITH THE INCOMPLETE H AND \bar{H} -FUNCTIONS AND ITS APPLICATIONS

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Abstract: In this work, we establish some Euler-type integrals formulas associated Legendre function, incomplete H -function and incomplete \bar{H} -function. Next, we present some interesting corollaries by specializing the parameters of IHF in the form of simpler special functions like the incomplete Meijer ${}^{(\Gamma)}G$ -function, incomplete Wright hypergeometric function, generalized-Mittag-Leffler function.

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1. Introduction

Several integral formulas associated with the various kind of special functions have been developed by many authors [1-5,7-12] and such type of integral formulas have been played an important role in the potential field of applied sciences, solving scientific and engineering problems. The main aim of this paper is to develop finite integral formulas containing the product of Legendre function, incomplete H-function and incomplete \bar{H} -function.

2. Preliminaries

We recall here classical definition of incomplete Gamma functions and $\gamma(a, x)$ defined by:

$$\gamma(a, x) = \int_0^x e^{-t} t^{a-1} dt, (\Re(a) > 0; x \geq 0), \quad (1)$$

and

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{a-1} dt, \quad (\Re(a) > 0; x \geq 0 \text{ if } x = 0). \quad (2)$$

The incomplete Gamma functions $\Gamma(a, x)$ and $\gamma(a, x)$ are satisfying decomposition formula given by:

$$\gamma(a, x) + \Gamma(a, x) = \Gamma(a), \quad (\Re(a) > 0). \quad (3)$$

Recently, Srivastava et.al [12] introduced the incomplete H-functions as follows:

$$\Gamma_{p,q}^{m,n}(z) = \Gamma_{p,q}^{m,n} \left(z \left| \begin{array}{c} (a_1, A_1, y), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array} \right. \right) = \frac{1}{2\pi i} \int_L G(\vartheta, y) z^{-\vartheta} d\vartheta, \quad (4)$$

where

$$G(\vartheta, y) = \frac{\Gamma(1-a_1-A_1\vartheta, y) \prod_{j=1}^m \Gamma(b_j+B_j\vartheta) \prod_{j=2}^n \Gamma(1-a_j-A_j\vartheta)}{\prod_{j=m+1}^q \Gamma(1-b_j-B_j\vartheta) \prod_{j=n+1}^p \Gamma(a_j+A_j\vartheta)}, \quad (5)$$

and

$$\gamma_{p,q}^{m,n}(z) = \gamma_{p,q}^{m,n} \left(z \left| \begin{array}{c} (a_1, A_1, y), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{array} \right. \right) = \frac{1}{2\pi i} \int_L g(\vartheta, y) z^{-\vartheta} d\vartheta, \quad (6)$$

where

$$g(\vartheta, y) = \frac{\gamma(1-a_1-A_1\vartheta, y) \prod_{j=1}^m \Gamma(b_j+B_j\vartheta) \prod_{j=2}^n \Gamma(1-a_j-A_j\vartheta)}{\prod_{j=m+1}^q \Gamma(1-b_j-B_j\vartheta) \prod_{j=n+1}^p \Gamma(a_j+A_j\vartheta)}. \quad (7)$$

The incomplete H-functions defined in (4) and (6) exist for all $y \geq 0$ under the set of conditions stated below. The path of the integral L in the complex ϑ -plane runs from $c - i\infty$ to $c + i\infty$, $c \in \mathbb{R}$, and singularity (poles) of the gamma functions $\Gamma(1 - a_j - A_j\vartheta)$, $j = \overline{1, n}$ do not exactly match with the singularity of the gamma functions $\Gamma(b_j + B_j\vartheta)$, $j = \overline{1, m}$. The parameters p and q are non-negative integers satisfying the relations $0 \leq n \leq p, 0 \leq m \leq q$ and the coefficients A_j, B_j ($j = 1, \dots, q$) $\in \mathbb{R}^+$ and a_j, b_j all are complex. All poles of $G(\vartheta, y)$ and $g(\vartheta, y)$ are supposed to be simple, and the empty product is treated as unity. The incomplete H-functions converges absolutely and define the analytic function for $|\arg(z)| < \frac{1}{2}\Omega\pi$, where,

$$\Omega = \sum_{j=1}^m B_j - \sum_{j=m+1}^q B_j + \sum_{j=1}^n A_j - \sum_{j=n+1}^p A_j.$$

The generalization of the incomplete H-function introduced and defined by Srivastava et al. [12] and it is also known as incomplete and incomplete \overline{H} -function.

$$\bar{\Gamma}_{p,q}^{m,n}(z) = \bar{\Gamma}_{p,q}^{m,n} \left(z \left| \begin{matrix} (a_1, A_1, \zeta_1, \gamma), (a_j, A_j, \zeta_j)_{2,n} (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m} (b_j, B_j, \eta_j)_{m+1,q} \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_L \bar{G}(\vartheta, \gamma) z^{-\vartheta} d\vartheta \tag{8}$$

where

$$\bar{G}(\vartheta, \gamma) = \frac{[\Gamma(1-a_1-A_1\vartheta, \gamma)]^{\zeta_1} \prod_{j=1}^m \Gamma(b_j+B_j\vartheta) \prod_{j=2}^n [\Gamma(1-a_j-A_j\vartheta)]^{\zeta_1}}{\prod_{j=m+1}^q [\Gamma(1-b_j+B_j\vartheta)]^{\eta_1} \prod_{j=n+1}^p \Gamma(a_j+A_j\vartheta)} \tag{9}$$

$$\bar{\gamma}_{p,q}^{m,n}(z) = \bar{\gamma}_{p,q}^{m,n} \left(z \left| \begin{matrix} (a_1, A_1, \zeta_1, \gamma), (a_j, A_j, \zeta_j)_{2,n} (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m} (b_j, B_j, \eta_j)_{m+1,q} \end{matrix} \right. \right) = \frac{1}{2\pi i} \int_L \bar{g}(\vartheta, \gamma) z^{-\vartheta} d\vartheta \tag{10}$$

$$\bar{g}(\vartheta, \gamma) = \frac{[\Gamma\gamma(1-a_1-A_1\vartheta, \gamma)]^{\zeta_1} \prod_{j=1}^m \Gamma(b_j+B_j\vartheta) \prod_{j=2}^n [\Gamma(1-a_j-A_j\vartheta)]^{\zeta_1}}{\prod_{j=m+1}^q [\Gamma(1-b_j+B_j\vartheta)]^{\eta_1} \prod_{j=n+1}^p \Gamma(a_j+A_j\vartheta)} \tag{11}$$

It will be assumed that all the incomplete \bar{H} -function occurring herein satisfy conditions of the existence corresponding appropriately to those given by [12] and $|\arg(z)| < \frac{1}{2}\bar{\Omega}\pi$ and $\bar{\Omega} = \sum_{j=1}^m |B_j| - \sum_{j=m+1}^q |\eta_j B_j| + \sum_{j=1}^n |\zeta_j A_j| - \sum_{j=n+1}^p |A_j| < \frac{\pi\bar{\Omega}}{2}$.

Currently several author work on a wide variety of applications for these incomplete functions, see, for example, recent work [2-5,7-9,11] and references therein.

We also recall the integral formula associated with Legendre function [6, p. 772, Eq.(4)] is defined as:

$$\int_0^1 x^\sigma (1-x^2)^{-\mu/2} p_\nu^\mu(x) dx = \frac{2^{\mu-1} \Gamma(\frac{1+\sigma}{2}) \Gamma(\frac{1+\sigma}{2})}{\Gamma(1+\frac{\sigma}{2}-\frac{\nu}{2}) \Gamma(\frac{\sigma}{2}+\frac{\nu}{2}+\frac{3}{2})} \tag{12}$$

3. A Set of Results

In this section, we established several integral formulas involving associated Legendre function, incomplete H - function and incomplete \bar{H} -function.

Theorem 1: If $y \geq 0$ and $\Re(\vartheta) > 0, \Omega > 0, k > 0, |\arg z| < \frac{\pi\Omega}{2}, \Re(\mu) < 1$ and $\Re(\sigma) < -1$, then the following integral formula holds:

$$\begin{aligned}
& \int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) \Gamma_{p,q}^{m,n} \left(zx^k \left| \begin{matrix} (a_1, A_1, y), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right) dx \\
&= 2^{\mu-1} \Gamma_{p+2, q+2}^{m, n+2} \left(z \left| \begin{matrix} (a_1, A_1, y), \left(\frac{1-\sigma}{2}, \frac{k}{2}\right) (a_j, A_j)_{2,p} \\ \left(\frac{v+\mu-\sigma}{2}, \frac{k}{2}\right), \left(\frac{\mu-v-\sigma-1}{2}, \frac{k}{2}\right), (b_j, B_j)_{1,q} \end{matrix} \right. \right) \quad (13)
\end{aligned}$$

provided that the conditions of incomplete H-function in [5] are satisfies.

Proof: To prove the assertion (13), express the incomplete H-function in terms of the Mellin-Barnes-type integral with the help of (4), will get say Δ :

$$\Delta = \int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) \frac{1}{2\pi i} \int_L G(\vartheta, y) (zx^k)^{-\vartheta} d\vartheta dx$$

Further changing the order of integration and with the help of (12), we get

$$\Delta = \frac{1}{2\pi i} \int_L G(\vartheta, y) (z)^{-\vartheta} \frac{2^{\mu-1} \Gamma\left(\frac{1+\sigma-k\vartheta}{2}\right) \Gamma\left(1+\frac{\sigma-k\vartheta}{2}\right)}{\Gamma\left(1+\frac{\sigma-v-\mu-k\vartheta}{2}\right) \Gamma\left(\frac{\sigma+v-\mu+3-k\vartheta}{2}\right)} d\vartheta.$$

Finally, applying (5) in the above integral and we arrive at the desire result after little simplification.

Theorem 2: If $y \geq 0$ and $\Re(\vartheta) > 0, \Omega > 0, k > 0, |\arg z| < \frac{\pi\Omega}{2}, \Re(\mu) < 1$ and $\Re(\sigma) < -1$

then, the following integral holds:

$$\begin{aligned}
& \int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) \gamma_{p,q}^{m,n} \left(zx^k \left| \begin{matrix} (a_1, A_1, y), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right) dx \\
&= 2^{\mu-1} \gamma_{p+2, q+2}^{m, n+2} \left(z \left| \begin{matrix} (a_1, A_1, y), \left(\frac{1-\sigma}{2}, \frac{k}{2}\right) (a_j, A_j)_{2,p} \\ \left(\frac{v+\mu-\sigma}{2}, \frac{k}{2}\right), \left(\frac{\mu-v-\sigma-1}{2}, \frac{k}{2}\right), (b_j, B_j)_{1,q} \end{matrix} \right. \right) \quad (14)
\end{aligned}$$

Proof: To prove assertion (14) can be evaluated by proceeding on the similar lines to the proof of theorem 1, by making use of (6) & (7).

Theorem 3: If $\Re(\vartheta) > 0, \bar{\Omega} > 0, k > 0, |\arg z| < \frac{\pi\bar{\Omega}}{2}, \Re(\mu) < 1$ and $\Re\left(\sigma + k \frac{b_j}{B_j}\right) > -1$, then the following formula holds for $y \geq 0$.

$$\int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) \bar{\Gamma}_{p,q}^{m,n} \left(z x^k \left| \begin{matrix} (a_1, A_1, \varsigma, \gamma), (a_1, A_1, \varsigma_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,q}, (b_j, B_j, \eta_j)_{m+1,q} \end{matrix} \right. \right) dx$$

$$= 2^{\mu-1} \bar{\Gamma}_{p+2,q+2}^{m,n+2} \left(z \left| \begin{matrix} (a_1, A_1, \varsigma_1, \gamma), \left(\frac{1-\sigma}{2}, \frac{k}{2}, 1\right), \left(\frac{-\sigma}{2}, \frac{k}{2}, 1\right), (a_j, A_j, \varsigma_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ \left(\frac{v+\mu-\sigma}{2}, \frac{k}{2}; 1\right), \left(\frac{\mu-v-\sigma-1}{2}, \frac{k}{2}; 1\right), (b_j, B_j)_{1,q}, (b_j, B_j, \eta_j)_{m+1,q} \end{matrix} \right. \right) \quad (15)$$

Proof: To prove the Theorem 3, we begin with the left hand side of assertion (15), make use of Mellin- Barnes type integral in (8) which defines the incomplete H-function. Then we get (Δ):

$$\Delta = \int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) \frac{1}{2\pi i} \int_L \bar{G}(\vartheta, y) (zx^k)^{-\vartheta} d\vartheta dx.$$

Next we changing the order of integration and using the (12), we get

$$\Delta = \frac{1}{2\pi i} \int_L \bar{G}(\vartheta, y) (z)^{-\vartheta} \frac{2^{\mu-1} \Gamma\left(\frac{1+\sigma-k\vartheta}{2}\right) \Gamma\left(1+\frac{\sigma-k\vartheta}{2}\right)}{\Gamma\left(1+\frac{\sigma-v-\mu-k\vartheta}{2}\right) \Gamma\left(\frac{\sigma+v-\mu+3-k\vartheta}{2}\right)} d\vartheta.$$

Finally applying (9) in the above integral and we obtain the required result in right-hand-side of (15) after little simplification.

Theorem 4: If $y \geq 0$ and $\Re(\vartheta) > 0, \bar{\Omega} > 0, k > 0, |\arg z| < \frac{\pi\bar{\Omega}}{2}, \Re(\mu) < 1$ and $\Re\left(\sigma + k \frac{b_j}{B_j}\right) > -1$, then the following formula holds:

$$\int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) \bar{\gamma}_{p,q}^{m,n} \left(z x^k \left| \begin{matrix} (a_1, A_1, \varsigma, \gamma), (a_1, A_1, \varsigma_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ (b_j, B_j)_{1,m}, (b_j, B_j, \eta_j)_{m+1,q} \end{matrix} \right. \right) dx$$

$$= 2^{\mu-1} \bar{\gamma}_{p+2,q+2}^{m,n+2} \left(z \left| \begin{matrix} (a_1, A_1, \varsigma_1, \gamma), \left(\frac{1-\sigma}{2}, \frac{k}{2}, 1\right), \left(\frac{-\sigma}{2}, \frac{k}{2}, 1\right), (a_j, A_j, \varsigma_j)_{2,n}, (a_j, A_j)_{n+1,p} \\ \left(\frac{v+\mu-\sigma}{2}, \frac{k}{2}; 1\right), \left(\frac{\mu-v-\sigma-1}{2}, \frac{k}{2}; 1\right), (b_j, B_j)_{1,m}, (b_j, B_j, \eta_j)_{m+1,q} \end{matrix} \right. \right) \quad (16)$$

provided that the conditions of incomplete \bar{H} -function [5] are satisfies

Proof: The assertion (16) of the Theorem 4 can be established by the similar lines of the above theorem, by making use of (11) & (12).

4. Special Cases

In this section, we are given some interesting special cases of our main finding:

Corollary 1: If $y \geq 0$ and $\Re(\vartheta) > 0, \Omega > 0, k > 0, 2(m+n) > p+q, \Re(\mu) < 1, |\arg z| < \left(m+n - \frac{p}{q} - \frac{q}{2}\right)\pi$, then the following integral formula involving the incomplete Meijer ${}^{(\Gamma)}$ G-function as given below:

$$\begin{aligned} & \int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) {}^{(\Gamma)}G_{p,q}^{m,n} \left(zx^2 \left| \begin{matrix} (a_j, y), (a_j)_{2,p} \\ (b_j)_{1,q} \end{matrix} \right. \right) dx \\ &= 2^{\mu-1} {}^{(\Gamma)}G_{p+2,q+2}^{m,n+2} \left(z \left| \begin{matrix} (a_1; y), \left(\frac{1-\sigma}{2}\right), \left(\frac{-\sigma}{2}\right), (a_j)_{2,p} \\ \left(\frac{v+\mu-\sigma}{2}\right), \left(\frac{\mu-v-\sigma-1}{2}\right), (b_j)_{1,q} \end{matrix} \right. \right) \end{aligned} \quad (17)$$

Proof: Taking $i = 1, \dots, m=1, k=2, A_i=1$ and $B_j=1$ in the result (13), then the following result deduce in terms of incomplete Meijer ${}^{(\Gamma)}$ G-function.

Corollary 2: If $y \geq 0$ and $\Re(\vartheta) > 0, \Omega > 0, k > 0, |\arg z| < \pi \left(m+n - \frac{p}{q} - \frac{q}{2}\right), \Re(\mu) < 1$

then the following integral formula involving of the incomplete Fox-Wright-function ${}_p\psi_q^{(\Gamma)}$ [12] are as follows:

$$\begin{aligned} & \int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) {}_p\psi_q^{(\Gamma)} \left(zx^k \left| \begin{matrix} (a_j, A_1, y), (a_j, A_j)_{2,p} \\ (b_j, B_j)_{1,q} \end{matrix} \right. \right) dx \\ &= 2^{\mu-1} {}_{p+2}\psi_{q+2}^{(\Gamma)} \left(z \left| \begin{matrix} (a_1, A_1, y), \left(\frac{1-\sigma}{2}, 1\right), \left(\frac{-\sigma}{2}, 1\right), (a_j, A_j)_{2,p} \\ \left(\frac{v+\mu-\sigma}{2}, 1\right), \left(\frac{\mu-v-\sigma-1}{2}, 1\right), (b_j, B_j)_{1,q} \end{matrix} \right. \right) \end{aligned} \quad (18)$$

Proof: Again, on taking $m=1, n=p$ and replacing q with $q=1, k=2$ in theorem 1 and the suitable adjustment of the parameters, the incomplete H -function reduces to incomplete Fox-Wright ${}_p\Psi_q^{(\Gamma)}$ -function [12] also see [5].

Corollary 3 : If $y \geq 0$ and $\Re(\vartheta) > 0, \Omega > 0, k > 0, |\arg z| < \pi \left(m+n - \frac{p}{q} - \frac{q}{2}\right), \Re(\mu) < 1$

then the following integral formula involving of the generalized Mittag-Leffler function as follows:

$$\int_0^1 x^\sigma (1-x^2)^{-\mu/2} P_v^\mu(x) E_{\alpha,\beta}^\delta [zx^2] dx$$

$$= 2^{\mu-1} H_{3,4}^{1,3} \left(z \left| \begin{array}{l} (1-\delta, 1), \left(\frac{1-\sigma}{2}, 1\right), \left(\frac{-\sigma}{2}, 1\right) \\ (0, 1), (1-\beta, \alpha), \left(\frac{v+\mu-\sigma}{2}, 1\right), \left(\frac{\mu-v-\sigma-1}{2}\right) \end{array} \right. \right) \quad (19)$$

Proof: In the assertion (13), we take $A_i = 1$, $y=1$, $k=2$, $m = n = p = 1$ and $q = 2$ incomplete H-function convert into the generalized Mittag-Leffler function by small adjustment in parameters. (see[12]).

5. Conclusion

Several authors developed a large number of research article on the various type of integral formulas involving the various kind of special functions. In this regard, we established some integral formulas associated with the Legendre function, incomplete H-function and incomplete \bar{H} -function. Further, we point out some interesting special cases of these integrals which are believed to be new.

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