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FINSLERIAN HYPERSURFACES WITH GENERALIZED (α,β)-METRIC

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Abstract: The present article is destined to study the Finslerian hypersurfaces of a Finsler space with the generalized (α,β)-metric such as $F=\nu_1\alpha+\nu_2\beta+\nu_3\dfrac{\beta^2}{\alpha+\beta}$ where ν_1, ν_2 and ν_3 are constants and $\nu_1\neq 0$. Also, we have looked at the hypersurfaces of this specific (α,β)-metric as hyperplane of first and second kind but not hyperplane of third kind.

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1. Introduction

Let a Finsler space $F^n=(M^n,F)$ defined on n-dimensional, i.e., a pair of an n-dimensional differential manifold M^n and a fundamental function $F(x,y)$. M. Matsumoto [8] developed the notion of the (α,β)-metric, which has been investigated by numerous authors including Shibata and others ([2], [3], [5], [11], [12]). If F is a positively homogeneous function of α and β of degree one, a Finsler metric $F(x,y)$ is termed as a (α,β)-metric $F(\alpha,\beta)$, where Riemannian metric $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ and one-form $\beta=b_i(x)y^i$ defined on M^n. A hypersurface M^{n-1} of the M^n denoted by the equality $x^i=x^i(u^\alpha)$, $\alpha:=1,2,\cdots,n-1$, here u^α are Gaussian co-ordinates on M^{n-1}.

Matsumoto [9] introduced the systematic theory of hypersurfaces in 1985. He studied numerous types of Finslerian hypersurfaces known as hyperplane of the first, second, and third kind. I.Y-Lee [6] discovered some Finslerian hypersurface findings with metric

$\alpha+\frac{\beta^2}{\alpha}$ in 2001. In 2009, Narasimhamurthy [10] have obtained some results of hypersurface of a Finsler space with the metric $\alpha+\frac{\beta^{n+1}}{\alpha^n}$. Chaubey [1] have obtained some geometrical properties of Finslerian hypersurfaces with special (α,β)-metric in 2017. Recently in [13], they proved that the hypersurface of a Finsler space with exponential (α,β)-metric as a hyperplane of first, second and third kinds. In 2020, Pradeep Kumar et al. [11] have investigated cetrain geometrical properties of hypersurface of a Finsler space with generalised (α,β)-metric, also proved this hypersurface as a hyperplane of first and second kind.

In this article, we investigated the cetrain geometrical properties of a hypersurface of a Finsler space with generalised (α,β)-metric $F=\nu_1\alpha+\nu_2\beta+\nu_3\frac{\beta^2}{\alpha+\beta}$ where ν_1,ν_2 and ν_3 are constants and $\nu_1\neq 0$. In addition, we have demonstrated that this hypersurface is a first and second kind hyperplane. M. Matsumoto [7] is cited for the notations and terminology used in this article. Firstly, we gave a introduction for Finslerian hypersurface in section one. In section two, we go through some basic concepts of fundamental metric tensor, cartan connections and hypersurface. In section three, we have discussed the criteria of induced Cartan connection. Finally, in section four we have discussed the hypersurfaces of a hyperplane of first, second and third kind with generalized (α,β)-metric.

2. Preliminaries

In this research article we have consider a $F^n=(M^n,F(\alpha,\beta))$ be a special Finsler space with the generalized (α,β)-metric

$$F=\nu_1\alpha+\nu_2\beta+\nu_3\frac{\beta^2}{\alpha+\beta}, \tag{1}$$

where ν_1, ν_2 and ν_3 are constants and $\nu_1\neq 0$. Partially differentiating equation (1) with respect to β and α are given by

$$\begin{aligned} F_\alpha &= \nu_1-\nu_3\frac{\beta^2}{(\alpha+\beta)^2}, \quad F_{\alpha\alpha} = 2\nu_3\frac{\beta^2}{(\alpha+\beta)^3}, \quad F_\beta = \nu_2+\nu_3\frac{2\alpha\beta+\beta^2}{(\alpha+\beta)^2}, \\ F_{\beta\beta} &= 2\nu_3\frac{\alpha^2}{(\alpha+\beta)^3}, \quad F_{\alpha\beta} = -2\nu_3\frac{\alpha\beta}{(\alpha+\beta)^3}. \end{aligned} \tag{2}$$

Where $F_\alpha = \frac{\partial F}{\partial \alpha}$, $F_{\alpha\alpha} = \frac{\partial F_\alpha}{\partial \alpha}$, $F_\beta = \frac{\partial F}{\partial \beta}$, $F_{\beta\beta} = \frac{\partial F_\beta}{\partial \beta}$ and $F_{\alpha\beta} = \frac{\partial F_\alpha}{\partial \beta}$.

If $F^n = (M^n, F(\alpha,\beta))$ be Finsler space the auxiliary element $l_i = \dot{\partial}_i F$ and angular-metric tensor h_{ij} are determined by [9]

$$l_i = F_\alpha \frac{y_i}{\alpha} + F_\beta b_i,$$
$$h_{ij} = p a_{ij} + p_1 b_i b_j + p_2 (b_j Y_i + b_i Y_j) + p_3 Y_i Y_j,$$

where

$$Y_i = a_{ij} y^j, \quad p = \frac{FF_\alpha}{\alpha}, \quad p_1 = FF_{\beta\beta}, \quad p_2 = \frac{FF_{\alpha\beta}}{\alpha}, \quad p_3 = \frac{F(F_{\alpha\alpha} - F_\alpha \alpha^{-1})}{\alpha^2}. \tag{3}$$

Using (2), equation (3) becomes

$$p = \frac{\left(\nu_1\alpha^2 + (\nu_1 - \nu_3)\beta^2 + 2\nu_1\alpha\beta\right)\left(\nu_1\alpha^2 + (\nu_2+\nu_3)\beta^2 + (\nu_1+\nu_2)\alpha\beta\right)}{\alpha(\alpha+\beta)^3},$$

$$p_1 = \frac{2\nu_3\alpha^2\left(\nu_1\alpha^2 + (\nu_2+\nu_3)\beta^2 + (\nu_1+\nu_2)\alpha\beta\right)}{(\alpha+\beta)^4},$$

$$p_2 = -\frac{2\nu_3\beta\left(\nu_1\alpha^2 + (\nu_2+\nu_3)\beta^2 + (\nu_1+\nu_2)\alpha\beta\right)}{(\alpha+\beta)^4},$$

$$p_3 = -\frac{\left(\nu_1\alpha^3 + (\nu_1-\nu_3)(\beta^3 + 3\alpha\beta^2 + 3\nu_1\alpha^2\beta)\right)\left(\nu_1\alpha^2 + (\nu_2+\nu_3)\beta^2 + (\nu_1+\nu_2)\alpha\beta\right)}{\alpha^3(\alpha+\beta)^4}. \tag{4}$$

The fundamental metric tensor $g_{ij} = \frac{\dot{\partial}_i \dot{\partial}_j F^2}{2}$, as well as it's reciprocal tensor g^{ij} for $F = F(\alpha,\beta)$ are given by [9]

$$g_{ij} = q a_{ij} + q_1 b_i b_j + q_2 Y_i Y_j + q_3\left(b_i Y_j + b_j Y_i\right),$$

where

$$q = p, \quad q_1 = p_1 + F_\beta^2, \quad q_2 = q_3 + \frac{p^2}{F^2}, \quad q_3 = p_2 + \frac{pF_\beta}{F}. \tag{5}$$

Using (2) and (4), equation (5) becomes

$$
\begin{aligned}
q_1 &= \frac{1}{(\alpha+\beta)^4}\Big[(2\nu_1\nu_3+\nu_2^2)\alpha^4+(2\nu_2+2\nu_3)^2\alpha\beta^3+(6\nu_3^2+12\nu_2\nu_3+6\nu_2^2)\alpha^2\beta^2+(4\nu_2^2\\
&\quad +2\nu_1\nu_3+6\nu_2\nu_3)\alpha^3\beta+(\nu_2+\nu_3)^2\beta^4\Big],\\
q_2 &= \frac{1}{\alpha(\alpha+\beta)^4}\Big[\nu_1\nu_2\alpha^4+(4\nu_1(\nu_3+\nu_2)-4\nu_3^2-4\nu_2\nu_3)\alpha\beta^3+(3\nu_1\nu_3+6\nu_1\nu_2-3\nu_2\nu_3)\\
&\quad \alpha^2\beta^2+4\nu_1\nu_2\alpha^3\beta+(\nu_1\nu_3-\nu_3^2-\nu_2\nu_3+\nu_1\nu_2)\beta^4\Big],\\
q_3 &= \frac{1}{\alpha^3(\alpha+\beta)^4}\Big[\nu_1\nu_2\alpha^4+(4\nu_1(\nu_3+\nu_2)-4\nu_3^2-4\nu_2\nu_3)\alpha\beta^3+(3\nu_1\nu_3+6\nu_1\nu_2-3\nu_2\nu_3)\\
&\quad \alpha^2\beta^2+4\nu_1\nu_2\alpha^3\beta+(\nu_1\nu_3-\nu_3^2-\nu_2\nu_3+\nu_1\nu_2)\beta^4\Big].
\end{aligned}
\tag{6}
$$

The reciprocal metric tensor g^{ij} of g_{ij} is given by

$$g^{ij} = xa^{ij} - x_1 b^i b^j - x_2\left(b^j y^i + b^i y^j\right) - x_3 y^i y^j,$$

where $b^i = a^{ij} b_j$ and $b^2 = a_{ij} b^i b^j$

$$
\begin{aligned}
x &= p^{-1}, \quad x_1 = \frac{1}{\tau q}\left\{qq_1 + \left(q_1q_3 - q_2^2\right)\alpha^2\right\}, \quad x_2 = \frac{1}{\tau q}\left\{qq_2 + \left(q_1q_3 - q_2^2\right)\beta\right\},\\
x_3 &= \frac{1}{\tau q}\left\{qq_3 + \left(q_1q_3 - q_2^2\right)b^2\right\}, \quad \tau = \left(q_1q_3 - q_2^2\right)\left(\alpha^2 b^2 - \beta^2\right) + q\left(q + q_1 b^2 + q_2\beta\right).
\end{aligned}
\tag{7}
$$

The hv-torsion tensor $C_{ijk} = \dfrac{\dot{\partial}_k g_{ij}}{2}$ is given by

$$pC_{ijk} = \frac{1}{2}\Big[\gamma_1 m_i m_j m_k + q_2\left(h_{ij} m_k + h_{jk} m_i + h_{ki} m_j\right)\Big], \tag{8}$$

where

$$\gamma_1 = q\frac{\partial q_1}{\partial\beta} - 3q_2 q_1, \quad m_i = b_i - \alpha^{-2}\beta Y_i. \tag{9}$$

In this case, m_i denotes a non vanishing co-variant vector orthogonal to the auxiliary element y^i. Let $\left\{\begin{matrix} i \\ jk \end{matrix}\right\}$ be the component of Christoffel symbols of the attributed Riemannian space R^n and ∇_k be the co-variant derivative with respect to x^k concerning this Christoffel symbol. Now we define,

$$2F_{ij} = b_{ij} - b_{ji}, \quad 2E_{ij} = b_{ij} + b_{ji}, \tag{10}$$

where $b_{ij} = \nabla_j b_i$.

Let $C\Gamma = \left(\Gamma^i_{jk}, \Gamma^{*i}_{0k}, \Gamma^{*i}_{jk}\right)$ be the Cartan-connection of Finsler space F^n. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \left\{ \begin{matrix} i \\ jk \end{matrix} \right\}$ of the specific Finsler space F^n is given by

$$\begin{aligned} D^i_{jk} = {} & F^i_k B_j + B^i E_{jk} + F^i_j B_k + B^i_k b_{0j} + B^i_j b_{0k} - b_{0m} g^{im} B_{jk} - C^i_{km} A^m_j - C^i_{jm} A^m_k + C_{jkm} A^m_s g^{is} \\ & + \lambda^s \left(C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms} \right), \end{aligned} \tag{11}$$

where

$$\begin{aligned} B_k &= q_1 b_k + q_2 Y_k, \quad F^k_i = g^{kj} F_{ji}, \quad B^i = g^{ij} B_j, \\ B_{ij} &= \frac{1}{2}\left[\frac{\partial q_1}{\partial \beta} m_i m_j + q_2 \left(a_{ij} - \alpha^{-2} Y_i Y_j \right) \right], \quad B^k_i = g^{kj} B_{ji}, \quad B_0 = B_i y^i, \\ \lambda^m &= B^m E_{00} + 2 B_0 F^m_0, \quad A^m_k = B^m_k E_{00} + B^m E_{k0} + B_k F^m_0 + B_0 F^m_k. \end{aligned} \tag{12}$$

Suffix '0' represents the contraction by the auxiliary-element y^i with the exception of the quantities p_1, q_1 and x_1.

3. Induced Cartan Connection

Let F^{n-1} be a hypersurface of Finsler space F^n denoted by the relation $x^i = x^i\left(u^\alpha\right)$, $\alpha := 1, 2, \ldots, n-1$, here u^α represents the Gaussian co-ordinates on the hypersurface M^{n-1}. Suppose that the projection factor matrix is $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ is of $(n-1)$-rank. If the auxiliary-element y^i at a certain point $u = \left(u^\alpha\right)$ of F^n is taken to be tangential to F^{n-1}, then $y^i = B^i_\alpha(u) v^\alpha$ consequently $v = \left(v^\alpha\right)$ is regarded as auxiliary-element of F^{n-1} at a certain point u^α. We get $(n-1)$-dimensional Finsler space $F^{n-1} = \left(F^{n-1}, \underline{F}(u,v)\right)$ because $\underline{F}(u,v) = F(x(u), y(u,v))$ gives rise to a Finslerian metric on F^{n-1}. The metric and Carton tensors $g_{\alpha\beta}$ and $C_{\alpha\beta\gamma}$ respectively are given by

$$C_{\alpha\beta\gamma} = C_{ijk} B^i_\alpha B^j_\beta B^k_\gamma, \quad g_{\alpha\beta} = g_{ij} B^i_\alpha B^j_\beta. \tag{13}$$

The typical unit $N^i(u,v)$ at every point u^α of F^{n-1} is characterised with

$$g_{ij} N^i N^j = 1, \quad g_{ij} B^i_\alpha N^j = 0. \tag{14}$$

For h_{ij} be the angular metric tensor, we have

$$h_{ij}B^i_\alpha N^j = 0, \quad h_{ij}N^i N^j = 1, \quad h_{\alpha\beta} = h_{ij}B^i_\alpha B^j_\beta. \tag{15}$$

The factor of inverse projection $B^\alpha_i(u,v)$ of B^i_α is determined as

$$B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \tag{16}$$

where $g^{\alpha\beta}$ is the inverse of $g_{\alpha\beta}$ of F^{n-1}. From (14) and (16), we have

$$N^i B^\alpha_i = 0, \quad N^i N_i = 1, \quad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0. \tag{17}$$

Furthermore, we have

$$N^i N_j + B^i_\alpha B^\alpha_j = \delta^i_j. \tag{18}$$

The normal curvature vector H_α and $H_{\alpha\beta}$ be the 2^{nd} fundamental h-tensor of F^{n-1} are given by

$$H_\alpha = N_i\left(B^i_{0\alpha} + G^i_j B^j_\alpha\right) \tag{19}$$

and

$$H_{\alpha\beta} = N_i\left(B^i_{\alpha\beta} + F^i_{jk}B^j_\alpha B^k_\beta\right) + M_\alpha H_\beta, \tag{20}$$

where $B^i_{\alpha\beta} = \dfrac{\partial x^i}{\partial u^\alpha \partial u^\beta}$, $M_\alpha = C_{ijk}B^i_\alpha N^j N^k$ and $B^i_{0\alpha} = B^i_{\beta\alpha}v^\beta$. It is clear that $H_{\alpha\beta}$ is not symmetric and

$$H_{\alpha\beta} - H_{\beta\alpha} = -M_\beta H_\alpha + M_\alpha H_\beta. \tag{21}$$

The equations (19) and (20) yields

$$H_{\alpha 0} = H_{\alpha\beta}v^\beta = H_\alpha + M_\alpha H_0, \quad H_{0\alpha} = H_{\beta\alpha}v^\beta = H_\alpha. \tag{22}$$

The $M_{\alpha\beta}$ is given by

$$M_{\alpha\beta} = C_{ijk}B^i_\alpha B^j_\beta N^k. \tag{23}$$

The derivatives of N^i and B^i_α are as follows

$$B^i_{\alpha|\beta} = M_{\alpha\beta}N^i, \quad B^i_{\alpha|\beta} = H_{\alpha\beta}N^i, \quad N^i_{|\beta} = -M_{\alpha\beta}B^\alpha_j g^{ij}, \quad N^i_{|\beta} = -H_{\alpha\beta}B^\alpha_j g^{ij}. \tag{24}$$

The h and v-covariant derivatives of vector field X_i are

$$X_{i|\beta} = X_{i|j}B^j_\beta, \quad X_{i|\beta} = X_{i|j}B^j_\beta + X_{i|j}N^jH_\beta. \tag{25}$$

In the next section, we will apply the following lemmas are given by Matsumoto [9].

Lemma 3.1 *Let* F^{n-1} *be a hypersurface is a hyperplane of the first kind if and only if* $H_\alpha = 0$ *or identically* $H_0 = 0.$

Lemma 3.2 *Let* F^{n-1} *be a hypersurface is a hyperplane of the second kind if and only if* $H_{\alpha\beta} = 0$

Lemma 3.3 *Let* F^{n-1} *be a hypersurface is a hyperplane of the third kind if and only if* $H_{\alpha\beta} = 0 = M_{\alpha\beta}$.

4. Finslerian Hypersurface with Special (α,β) -metric $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha+\beta}$

Consider a Finsler space with generalised (α,β) -metric $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha+\beta}$, where ν_1, ν_2 and ν_3 are constants and $\nu_1 \neq 0$, Riemannian metric $\alpha^2 = a_{ij}(x)y^iy^j$, one-form $\beta = b_i(x)y^i$ and vector field $b_i(x) = \dfrac{\partial b}{\partial x^i}$ is a gradient of scalar-function $b(x)$. Next, we consider $F^{n-1}(c)$ be a hypersurface given by $b(x) = c$ [9].

Using the equation of parametric $x^i = x^i(u^\alpha)$ of $F^{n-1}(c)$, we have

$$\frac{\partial b(x)}{\partial u^\alpha} = 0, \quad \frac{\partial b(x)}{\partial x^i}\frac{\partial x^i}{\partial u^\alpha} = 0, \quad b_iB^i_\alpha = 0.$$

This shows that $b_i(x)$ is co-variant component of a regular vector field of $F^{n-1}(c)$. Furthermore, we have

$$b_iB^i_\alpha = 0 \quad and \quad b_iy^i = 0 \quad i.e., \beta = 0. \tag{26}$$

The induced-metric $\bar{F}(u,v)$ of $F^{n-1}(c)$ which is Riemannain metric as follows

$$\bar{F}(u,v) = a_{\alpha\beta}, \quad a_{\alpha\beta} = a_{ij}B^i_\alpha B^j_\beta. \tag{27}$$

Considering $\beta = 0$ in the equation (4), (6) and (7) we get

$$p = \nu_1^2, \quad p_1 = 2\nu_1\nu_3, \quad p_2 = 0, \quad p_3 = -\frac{\nu_1^2}{\alpha^2}, \quad q = \nu_1^2, \quad q_1 = \nu_2^2 + 2\nu_1\nu_3, \quad q_2 = \frac{\nu_1\nu_2}{\alpha},$$
$$p_3 = 0, \quad x = \frac{1}{\nu_1^2}, \quad x_1 = \frac{2\nu_3}{\nu_1^2(2\nu_3 b^2 + \nu_1)}, \quad x_2 = \frac{\nu_2}{\nu_1^2\alpha(2\nu_3 b^2 + \nu_1)}, \quad \tau = \nu_1^3(2\nu_3 b^2 + \nu_1) \tag{28}$$
$$x_3 = -\frac{b^2\nu_2^2}{\nu_1^3\alpha^2(2\nu_3 b^2 + \nu_1)}.$$

Reciprocal of fundamental metric tensor g^{ij} becomes

$$g^{ij} = \frac{a^{ij}}{\nu_1^2} + \frac{2\nu_3}{\nu_1^2(2\nu_3 b^2 + \nu_1)} b^i b^j + \frac{\nu_2}{\nu_1^2\alpha(2\nu_3 b^2 + \nu_1)}(b^i y^j + b^j y^i) - \frac{b^2\nu_2^2}{\nu_1^3\alpha^2(2\nu_3 b^2 + \nu_1)} y^i y^j. \tag{29}$$

Hence along $F^{n-1}(c)$, (26) and (29) yields

$$g^{ij} b_i b_j = \frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}.$$

So we get

$$b_i(x) = \sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} N_i, \quad b^2 = a^{ij} b_i b_j, \tag{30}$$

where the length of the vector b^i is denoted by b.

Using (29) and (30), we have

$$b^i = a^{ij} b_j = \sqrt{\nu_1^3 b^2(2\nu_3 b^2 + \nu_1)} N^i + \frac{\nu_2 b^2}{\alpha} y^i. \tag{31}$$

Hence, we have the following

Theorem 4.1 *The generalised* (α, β) *-metric* $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha + \beta}$ *on a Finsler hypersurface* $F^{n-1}(c)$*, the induced-metric is advanced by equation (27) and* $b(x)$ *is a scalar function is given by (30) and (31).*

Now h_{ij} and g_{ij} of Finsler space F^n are given by

$$h_{ij} = \nu_1^2 a_{ij} + 2\nu_1\nu_2 b_i b_j - \frac{\nu_1^2}{\alpha^2} y_i y_j \tag{32}$$

and

$$g_{ij} = \nu_1^2 a_{ij} + 2(\nu_2^2 + 2\nu_1\nu_2)b_i b_j + \frac{\nu_1\nu_2}{\alpha}(b_i y_j + b_j y_i).$$

If $h^{(a)}_{\alpha\beta}$ a angular metric tensor of $a_{ij}(x)$ be the Riemannian tensor by using (26), (32) and (15), then along with $F^{n-1}(c)$,

$$h_{\alpha\beta} = h^{(a)}_{\alpha\beta}.$$

From (6), we get

$$\frac{\partial q}{\partial \beta} = \frac{6\nu_3\alpha^3\left(\alpha+\beta-\nu_1(\alpha+\beta)+2\nu_3\beta+\nu_2(\alpha+\beta)\right)}{(\alpha+\beta)^5}.$$

Thus along $F^{n-1}(c)$, $\dfrac{\partial q}{\partial \beta} = \dfrac{6}{\alpha}\left[\nu_3 - \nu_1\nu_3 + \nu_2\nu_3\right]$.

Therefore equation (9) gives $\gamma_1 = \dfrac{6}{\alpha}\left(\nu_3 - \nu_1\nu_3\right)$ *and* $m_i = b_i$, then hv-tensor C_{ijk} becomes

$$C_{ijk} = \frac{\nu_2}{2\nu_1\alpha}\left[h_{ij}b_k + h_{jk}b_i + h_{ki}b_j\right] + \frac{3}{\alpha}\left[\nu_3 - \nu_1\nu_3\right]b_i b_j b_k, \qquad (33)$$

in the $F(\alpha,\beta)$-metric of a hypersurface $F^{n-1}(c)$. Therefore from (15), (23), (26) and (33), we get

$$M_{\alpha\beta} = \frac{\nu_2}{2\nu_1\alpha}\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}}\,h_{\alpha\beta} \quad and \quad M_\alpha = 0. \qquad (34)$$

Hence from (25) we have $H_{\alpha\beta} = H_{\beta\alpha}$, that is, $H_{\alpha\beta}$ is symmetric.

Thus, we have the following :

Theorem 4.2 *Let* F^{n-1} *be a* 2^{nd} *fundamental v-tensor of a Finsler space equipped with* (α,β) *-metric* $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha+\beta}$, *is advanced by (34) and* $H_{\alpha\beta}$ *be a* 2^{nd} *fundamental h-tensor is symmetric.*

Now from (26) we have $b_i B^i_\alpha = 0$. Then we have

$$b_{i|\beta}B^i_\alpha + b_i B^i_{\alpha|\beta} = 0. \qquad (35)$$

Applying (25) for the b_i vector, we have

$b_{i|\beta} = b_{i|j} B^j_\beta + b_{i|j} N^j H_\beta.$

Using $b_{i|\beta}$ and $B^i_{\alpha|\beta} = H_{\alpha\beta} N^i$, (35) becomes

$$b_{i|j} B^i_\alpha B^j_\beta + b_{i|j} B^i_\alpha N^j H_\beta + b_i H_{\alpha\beta} N^i = 0. \quad (36)$$

Since $b_{i|j} = -b_h C^h_{ij}$, we get $b_{i|j} B^i_\alpha N^i = 0$. Then using that $b_{i|j}$ is symmetric, from (36) we have

$$\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} H_{\alpha\beta} + b_{i|j} B^i_\alpha B^j_\beta = 0. \quad (37)$$

Now, contracting (37) with v^β, we get

$$\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} H_\alpha + b_{i|j} B^i_\alpha y^j = 0. \quad (38)$$

Again contracting (38) with v^α, we get

$$\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} H_0 + b_{i|j} y^i y^j = 0. \quad (39)$$

Based on lemma (3.1) that the Finslerian hypersurface $F^{n-1}(c)$ is a hyperplane of 1^{st} kind if and only if $H_0 = 0$. It is obvious that $F^{n-1}(c)$ is hyperplane of 1^{st} kind if and only if $b_{i|j} y^i y^j = 0$ is from equation (37). Here the covariant derivative $b_{i|j}$ depends on y^i, but $b_{ij} = \nabla_j b_i$ is the co-variant derivative related to Riemannian-connection $\left\{ {i \atop jk} \right\}$ constructed from $a_{ij}(x)$. Thus b_{ij} is unrelated to y^i.

Now, we considering the difference $b_{i|j} - b_{ij}$. The difference tensor $D^i_{jk} = \Gamma^{*i}_{jk} - \left\{ {i \atop jk} \right\}$ is given by (11). Since gradient vector b_i, then using (10) we have

$$F_{ij} = 0, \quad F^i_j = 0 \quad \text{and} \quad E_{ij} = b_{ij}.$$

Hence equation(11) reduces to the following

$$D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m} g^{im} B_{jk} - C^i_{jm} A^m_k - C^i_{km} A^m_j + C_{jkm} A^m_s g^{is} + \lambda^s \left(C^i_{jm} C^m_{sk} + C^i_{km} C^m_{sj} - C^m_{jk} C^i_{ms} \right), \tag{40}$$

By using (6) and (7), equation (12) becomes

$$B_k = \left(\nu_2^2 + 2\nu_1\nu_3\right) b^0 + \frac{\nu_1\nu_2}{\alpha} y^i, \quad B^i = \frac{2\nu_3}{\nu_1(2\nu_3 b^2 + \nu_1)} b^i + \frac{\nu_2}{\alpha(2\nu_3 b^2 + \nu_1)} y^i, \quad \lambda^m = B^m b_{00}$$
$$B_{ij} = \frac{\nu_1\nu_2}{\alpha}\left(a_{ij} - \frac{y_i y_j}{\alpha^2}\right) + \frac{6}{\alpha}\left(\nu_3(1-\nu_1+\nu_2)\right) b_i b_j, \quad B^i_j = g^{ik} B_{kj}, \quad A^m_k = B^m_k b_{00} + B^m b_{k0}. \tag{41}$$

Because of (26), we obtain $B_{i0} = 0$, $B^i_0 = 0$. This leads to $A^m_0 = B^m b_{00}$.

Next, contracting (40) by y^k, we have

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^m C^i_{jm} b_{00}.$$

Again contracting the above equation by y^j, we get

$$D^i_{00} = B^i b_{00} = \left[\frac{2\nu_3}{\nu_1(2\nu_3 b^2 + \nu_1)} b^i + \frac{\nu_2}{\alpha(2\nu_3 b^2 + \nu_1)} y^i \right] b_{00}.$$

Using the equation (26) along $F^{n-1}(c)$, we obtain

$$b_i D^i_{j0} = \frac{2\nu_3 b^2}{(\nu_1 + 2\nu_3 b^2)} b_{j0} + \frac{\nu_1\nu_2 + 6\left[\nu_3(1-\nu_1+\nu_2)\right] b^2}{2\alpha\nu_1(\nu_1 + 2\nu_3 b^2)} b_j b_{00} - \frac{2\nu_3}{\nu_1(\nu_1 + 2\nu_3 b^2)} b^m b_i C^i_{jm} b_{00}. \tag{42}$$

Now contracting (42) by y^i, we get

$$b_i D^1_{00} = \frac{2\nu_3 b^2}{(\nu_1 + 2\nu_3 b^2)} b_{00}. \tag{43}$$

Let $b_{i|j}$ be co-variant derivative of b_i with respect to x^i concerning the Cartan connections of F^n and the co-variant derivative of b_i with respect to x^j concerning the Riemannian connections is denoted by $b_{ij} = \Delta_j b_i$.

$$b_{i|j} - b_{ij} = \left(\partial_j b_i - F^r_{ij} b_r\right) - \left(\partial_j b_i - \begin{Bmatrix} r \\ jk \end{Bmatrix} b_r\right) = -\left(F^r_{ij} - \begin{Bmatrix} r \\ jk \end{Bmatrix}\right) b_r = -D^r_{ij} b_r,$$
$$b_{i|j} = b_{ij} - D^r_{ij} b_r. \tag{44}$$

From (20), (30), (31), (34) and $M_\alpha = 0$, we have $b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0.$

Then the equation $b_{i|j} = b_{ij} - D^r_{ij} b_r$, (42) and (43) gives $b_{i|j} y^i y^j = b_{00} - b_r D^r_{00} = \frac{\nu_3}{\nu_1(\nu_1 + 2\nu_3 b^2)} b_{00}.$

Consequently, (38) and (39) may be written as

$$\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} H_\alpha + \frac{\nu_3}{\nu_1(\nu_1 + 2\nu_3 b^2)} b_{i0} B^i_\alpha = 0,$$

$$\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} H_0 + \frac{\nu_3}{\nu_1(\nu_1 + 2\nu_3 b^2)} b_{00} = 0. \tag{45}$$

Hence, H_0 is related to $b_{00} = 0$, here b_{ij} is independent of y^i using the fact that $\beta = b_i y^i = 0$ the condition $b_{00} = 0$ can be written as $b_{ij} y^i y^j = b_i y^i b_j y^j$ for some $c_j(x)$, as a result

$$2b_{ij} = b_j c_i + b_i c_j. \tag{46}$$

Next, from (26) and (46), it follows that

$$b_{00} = 0, \quad b_{ij} B^i_\alpha B^j_\beta = 0, \quad b_{ij} B^i_\alpha y^j = 0.$$

Again (46) and (41) gives $b_{i0} b^i = \frac{c_0 b^2}{2}$, $\lambda^m = 0$, $A^i_j B^j_\beta = 0$ and $B_{ij} B^i_\alpha B^j_\beta = \frac{1}{2\alpha} h_{\alpha\beta}.$

Now using (20), (29), (31), (34) and (40), we can write

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = \frac{\nu_2 c_0 b^2}{4\nu_1^2 \alpha (2\nu_3 b^2 + \nu_1)^2} h_{\alpha\beta}.$$

Therefore in view of (44), equation (37) reduces to

$$\sqrt{\frac{b^2}{\nu_1(2\nu_3 b^2 + \nu_1)}} H_{\alpha\beta} + \frac{\nu_2 c_0 b^2}{4\nu_1^2 \alpha (2\nu_3 b^2 + \nu_1)^2} h_{\alpha\beta} = 0. \tag{47}$$

Theorem 4.3 *The necessary and sufficient condition for a* (α, β) *-metric* $F = \nu_1 \alpha + \nu_2 \beta + \nu_3 \frac{\beta^2}{\alpha + \beta}$ *in Finslerian hypersurface* $F^{n-1}(c)$ *to be a first kind* *hyperplane is given by equation (46). In this instance the 2 nd fundamental metric tensor of* $F^{n-1}(c)$ *is proportional to its angular-metric tensor.*

By lemma (3.2), $F^{n-1}(c)$ Finslerian hypersurface is second kind hyperplane if and only if $H_{\alpha\beta} = 0.$ Hence from equation (47) we obtained $c_0 = c_i(x)y^i = 0$. Thus $\exists$ $e(x)$ å $c_i(x) = e(x)b_i(x).$

Hence equation (46) yields

$$b_{ij} = e(x)b_i(x)b_j(x). \tag{48}$$

Theorem 4.4 *The necessary and sufficient condition for a* (α,β) *-metric* $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha+\beta}$ *in Finslerian hypersurface* $F^{n-1}(c)$ *to be a second kind hyperplane is given by equation (48).*

In view of (34) and lemma (3.3), we get

Theorem 4.5 *A* (α,β) *-metric* $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha+\beta}$ *in Finslerian hypersurface* $F^{n-1}(c)$ *is not a third kind hyperplane.*

5. Conclusion

In the present article, we have been examined the several types of Finslerian hypersurfaces of a Finsler space with the generalized (α,β) -metric such as $F = \nu_1\alpha + \nu_2\beta + \nu_3\dfrac{\beta^2}{\alpha+\beta}$ where ν_1, ν_2 and ν_3 are constants and $\nu_1 \neq 0$. Also we have obtained results for the hypersurfaces of this specific (α,β) -metric as hyperplane of first and second kind but not third kind.

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References

[1] Chaubey, V.K. and Mishra, A. (2017). Hypersurfaces of a Finsler space with a special (α, β)-metric, J. Contemp. Math. Anal., **52**(1), 1-7.

[2] Hashiguchi, M., Hojo, S., and Matsumoto, M. (1973). On Landsberg spaces of two dimensions with (α, β)-metric, J. Korean Math. Soc. **10**(1), 17-26.

[3] Kikuchi, and Shigetaka. (1979). On the condition that a space with (α, β)-metric be locally Minkowskian, Tensor (N, S.), **33**(2), 242-246.

[4] Lee, I.Y. Park, H.Y. and Lee Y.D. (2001). On a hypersurface of a special Finsler space with a metric $\alpha + \frac{\beta^2}{\alpha}$, Korean J. Math. Sciences. **8**(1), 93-101.

[5] Matsumoto, M. (1985). The induced and intrinsic Finsler connections of a hypersurface and Finslerien projective geometry, *J. Math*. Kyoto Univ. **25**(1), 107-144.

[6] Matsumoto, M. (1986). Foundations of Finsler geometry and special Finsler spaces, Kaiseisha.

[7] Matsumoto, M. (1992). Theory of Finsler spaces with (α, β)-metric, Rep. Math. Phys. **31**(1), 43-83.

[8] Narasimhamurthy, S.K. Aveesh, S.T. Nagaraja, H.G. and Pradeep Kumar (2009). On special hypersurface of a Finsler space with the metric $\alpha + \frac{\beta^{n+1}}{\alpha^n}$, Acta Univ. Apulensis Math. Inform. **17**, 129-139.

[9] Narasimhamurthy, S.K. Pradeep Kumar, and Aveesh, S.T. (2008). A study of hypersurfaces on special Finsler spaces, Int. J. Pure. Appl. Math. **48**(1), 67-74.

[10] Pradeep Kumar, and Madhu, T.S. (2020). On Finslerian hypersurface with generalized (α, β)-metric, Waffen-und Kostumkunde Journal, **11**(5), 55-68.

[11] Pradeep Kumar, Narasimhamurthy, S.K. Nagaraja, H.G. and Aveesh, S.T. (2009). On a special hypersurface of a Finsler space with (α, β)-metric, Tbil. Math. J. **2**, 51-60.

[12] Shibata, C. (1984). On Finsler spaces with an (α, β)-metric, J. Hokkaido Univ. Educ. **35**(1), 1-16.

[13] Tripathi, B.K. (2020). Hypersurfaces of a Finsler space with exponential form of (α, β)-metric, Ann. Univ. Craiova Math. Comput. Sci. Ser. **47**(1), 132-140.