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MATHEMATICAL ANALYSIS OF SOME VARIABLE PROPERTIES OF FLUID IN THIN FILM FLOW WITH HEAT SOURCE/SINK OVER PERMEABLE STRETCHING SHEET

R.S. Yadav[1], Sharad Sinha[2], O.D. Makinde[3] and Praveen Kumar[4]
[1,2,4]Department of Mathematics, University of Rajasthan,
Jaipur-302004, Rajasthan, India
[3]Faculty of Military Science, Stellenbosch University, South Africa
Email: [1]rajendrauor@gmail.com, [2]sharadsinha89@gmail.com
[3]makinded@gmail.com, [3]praveenydvuor79@gmail.com

Abstract: Flow of Magnetohydrodynamic (MHD) fluid in thin film over permeable stretching sheet with heat source/sink and some variable properties of fluid are proposed. The effects of viscous dissipation and Joule heating are also considered in the problem. The system of governing equations converted into a set of suitable ordinary differential equations. The resulting differential equations of the thin liquid film flow problem are cracked by shooting method with fourth order Runge-Kutta technique. The effects of various physical parameters like heat source/sink parameter, thermal conductivity parameter, Prandtl number, the viscosity parameter, magnetic parameter and Eckert number on velocity and temperature profiles are explained through graphs. The skin friction coefficient at the surface is presented through tables and same for the Nusselt number.

Keywords: Magnetohydrodynamics, Variable thermal conductivity, Thin film, Heat source/sink, Unsteady, Stretching sheet.

1. Introduction

Heat transfer and boundary layer on stretching sheet of thin liquid film flow has received attention from scientists. It has various applications in industrial and engineering processes. Grubka and Bobba [7] investigated boundary layer flow of viscous fluid on stretching sheet. Wang [20] explained about asymptotic results for thin and wide film. Andersson [2] analyzed those at large values of Prandtl number temperature of stretching surface became independent. Dandapat et al. [5] obtained that there is no thermocapillary symptoms in thin liquid film at large values of Prandtl number. Wang [19] found exact solution of thin liquid film flow. The problem analyzed by homotopy analysis method (HAM). Dandapat et al. [6] have extended his work for some variable fluid properties. Abel et al. [1] have studied liquid film flow with variable magnetic field by fourth order Runge-Kutta method. Makinde [12] obtained that a structure of minimum entropy generation can be created with correct selection and suitable combination of several thermophysical properties. Noor and Hashim [13] considered the effect of magnetic field. They analyzed impact of Hartmann number on thin liquid film flow by HAM. Aziz et al. [3] concluded by HAM that heat generation/absorption parameter has not considerable impact on velocity of thin liquid film flow over sheet. Liu and Megahed [10] found a drop in cooling amount for thin liquid film flow in shadow of internal heating and thermal radiation. Mahmoud [11] obtained that viscosity of the fluid flow had impact on skin friction coefficient. Pal et al. [14] studied radiation effect in detail on the flow of thin liquid film over permeable surface. Kumar et al. [9] investigated that film thickness parameter meet a tendency to reduce the fluid temperature. Yadav and Makinde [21] obtained effects of thermal radiation, joule heating and porous medium on the flow of a thin liquid film on an unsteady stretching sheet. Ushachew et al. [18] discussed numerical study of MHD heat convection of nanofluid in an open enclosure with internal heated objects and sinusoidal heated bottom. Sinha and Yadav [16] investigated MHD mixed convection slip flow along an inclined porous plate in presence of viscous dissipation and thermal radiation. Reddy and Makinde [15] explained numerical study on MHD radiating and reacting unsteady slip flow past a vertical permeable plate in a porous medium. Motivated by all the above works, aim of this research work is extending the work of Mahmoud and investigate effect of some variable fluid properties on thin liquid film with magnetic field and heat source/sink. Numerical analysis of the mathematical model of the problem is done by shooting method with fourth order Runge-Kutta technique.

2. Mathematical Formulation

We considered two dimensional incompressible viscous flows on a horizontal thin elastic sheet (Figure 1). The stretching velocity and surface temperature **[3]** of the sheet are given by

$$U(x,t) = \frac{bx}{(1-at)} \tag{1}$$

$$T_s(x,t) = T_0 - T_{ref}\left[dx^2/2\upsilon\right]\left(1-at\right)^{-3/2} \tag{2}$$

where a and b are constants. T_0 is the temperature of the fluid at the slot, T_{ref} is the constant reference temperature such that $0 \le T_{ref} \le T_0$, d is the constant of proportionality and υ is the kinematic viscosity.

The variable viscosity of the fluid and thermal conductivity **[6]** in terms of temperature are given by

$$\mu = \mu_0 e^{-\zeta\theta} \tag{3}$$

$$\kappa = \kappa_0\left(1 + c\theta\right) \tag{4}$$

where μ_0 and κ_0 are the fluid dynamic viscosity and thermal conductivity at temperature $T = T_0$ respectively, $\zeta = \zeta_0\left(T_s - T_0\right)$ is the expression of viscosity parameter, ζ_0 is the positive constant for fluid and c is the parameter of thermal conductivity. The fluid variable electrical conductivity is prescribed as $\sigma = \sigma_0 (1 - at)^{-1}$ where σ_0 is the fluid initial electrical conductivity at t = 0. The x-axis considered in the direction of motion of the sheet and the y-axis is chosen normal to it.

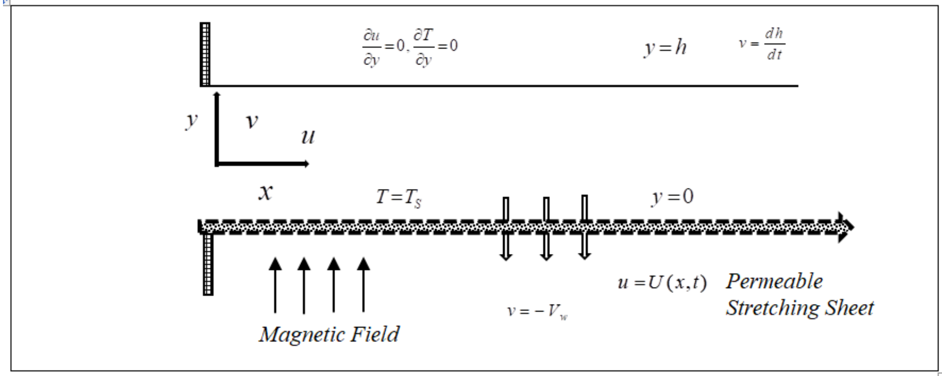

Figure 1. Physical sketch of the flow problem of thin liquid film.

The governing boundary layer flow and heat transfer partial differential equations **[17,19]** are given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) - \frac{\sigma B_0^2}{\rho}u, \tag{6}$$

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{1}{\rho C_p}\frac{\partial}{\partial y}\left(\kappa\frac{\partial T}{\partial y}\right) + \frac{\mu}{\rho C_p}\left(\frac{\partial u}{\partial y}\right)^2 + \frac{\sigma B_0^2}{\rho C_p}u^2 + \frac{1}{\rho C_p}Q \tag{7}$$

where Q is the heat source or sink **[11]** represented by

$$Q = \frac{\kappa U}{x\upsilon}(T - T_0)B^* \tag{8}$$

where B^* is the coefficient of heat source/sink. u is the velocity about the x axis and v is the velocity about y axis. $\upsilon\left(=\frac{\mu_0}{\rho}\right)$ is the kinematic viscosity, σ is the electrical conductivity, ρ is the density of the fluid, μ is the coefficient of viscosity, B_0 is the imposed uniform magnetic field strength, κ is the thermal conductivity of the fluid, C_p is the specific heat at constant pressure and T is fluid temperature.

The boundary conditions **[3]** are given by

$$y = 0: \; u = U(x,t), \quad v = -V_w, \quad T = T_s \quad (9)$$

$$y \to h: \; \frac{\partial u}{\partial y} = 0, \; \frac{\partial T}{\partial y} = 0, \; v = \frac{dh}{dt} \quad (10)$$

where h is considered the thickness of thin film, V_w is the suction velocity and T_s is the temperature of the stretching sheet. Introducing the similarity variable η and dimensionless functions f and θ **[5],** we get

$$\eta = (b/\upsilon)^{1/2}(1-at)^{-1/2}\xi^{-1}y \tag{11}$$

$$f(\eta) = \Psi(x,y,t)\left[\upsilon b x^2 \xi^{-2}/(1-at)\right]^{-1/2} \tag{12}$$

$$\theta(\eta) = -\frac{(T-T_0)}{T_{ref}}\left(\frac{2\upsilon}{dx^2}\right)(1-at)^{3/2} \tag{13}$$

$$u = \frac{\partial \Psi}{\partial y} = bx(1-at)^{-1} f'(\eta) \tag{14}$$

$$v = -\frac{\partial \Psi}{\partial x} = -(\upsilon b)^{1/2}(1-at)^{-1/2}\xi f(\eta) \tag{15}$$

where $\Psi(x,y,t)$ is the stream function. Substituting above similarity variables and dimensionless functions into (6) and (7), we obtain

$$(f''' - \zeta f''\theta') = e^{\zeta\theta}\xi^2\left[S\{f' + (\eta/2)\, f''\} + Mf' + (f')^2 - ff''\right] \tag{16}$$

$$\left[(1+c\,\theta)\theta'' + c(\theta')^2 + (1+c\,\theta)\xi^2 B^*\theta\right] = \Pr\xi^2\left[S\{(3/2)\theta + (\eta/2)\theta'\} + 2f'\theta - f\theta'\right] - e^{-\zeta\theta}\Pr Ec(f'')^2 - \xi^2 J(f')^2 \tag{17}$$

and the corresponding boundary conditions are reduced to

$$f(0) = f_w, \; f'(0) = 1, \; \theta(0) = 1, \; f(1) = S/2, \; f''(1) \to 0, \; \theta'(1) \to 0 \tag{18}$$

where prime indicates differentiation with respect to η, $M\left(=\frac{\sigma_0 B_0^2}{\rho b}\right)$ is the magnetic parameter due to transverse magnetic field, $\Pr\left(=\frac{\mu_0 C_p}{\kappa_0}\right)$ is the Prandtl number, $Ec\left(=\frac{U^2}{C_p(T_s - T_0)}\right)$ is the Eckert number, $J\left(=\frac{\sigma_0 B_0^2 U^2}{\rho b C_p (T_s - T_0)}\right)$ is the Joule heating parameter and $S\left(=\frac{a}{b}\right)$ is the unsteadiness parameter and $f_w\left(=\frac{hV_w}{\upsilon\xi^2}\right)$ is the permeability parameter.

The skin friction coefficient and Nusselt number of the fluid at the stretching sheet are represented by

$$C_f = \frac{2\tau_w}{\rho U^2} = -2\xi^{-1}\,\mathrm{Re}_x^{-1/2}\, e^{-\zeta\theta(0)} f''(0) \tag{19}$$

$$Nu = \frac{xq_w}{\kappa_0 T_{ref}} = -m\,\mathrm{Re}_x\big(1 + c\theta(0)\big)\theta'(0) \tag{20}$$

where $\mathrm{Re}_x = \frac{xU}{\upsilon}$ is the Reynolds number, $m = \frac{dU}{2\xi\upsilon^{1/2}b^{3/2}}$, $\tau_w = -\mu\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is the shearing stress and $q_w = -\kappa\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is the rate of heat transfer in the fluid at the stretching sheet.

3. Numerical Method

Since coupled differential equations (16) and (17) are highly non-linear in nature. Therefore, it is not feasible to occur during exact solution of the problem. In the present investigation, boundary value problem has been solved numerically [8] using Runge-Kutta algorithm with shooting technique for several values of physical parameters of Newtonian viscous thin liquid film flow. The convergence standards are based on the best shots of the unknown initial values.

4. Results and Discussion

It is revealed from figure 2 that the velocity of fluid leads a decreasing behavior towards increase in magnetic parameter. This is happened due to transverse magnetic field called Lorentz force. This force decelerated the fluid particles along the thin liquid film. It is obvious from figure 3 that the temperature of fluid increases with increasing value of magnetic parameter. Due to Lorentz force the width of thermal boundary layer rises. Hence the temperature of fluid increases.

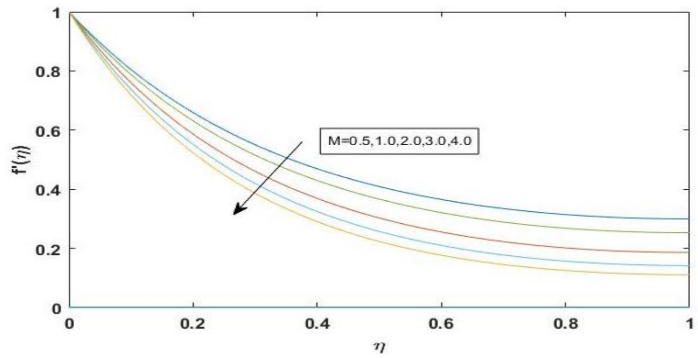

Figure 2. Velocity profiles of fluid flow versus **η** when $c = 0.2, S = 1.3, Ec = 0.5, Pr = 1.0, \zeta = 0.2, B^* = 0.5$.

It is noted from figure 4 that fluid velocity declines with enhancing values of unsteadiness parameter. In actual, the unsteadiness parameter decreases the convective potential between the thin liquid film and fluid, therefore the fluid velocity decreases. Figure 5 illustrates the behavior of unsteadiness parameter on temperature profile. It is obvious from figure 5 that the temperature of fluid decreases towards increasing values of unsteadiness parameter. This is happened because increasing in unsteadiness parameter reduces the temperature difference between thin liquid film and fluid and hence temperature of fluid decreases.

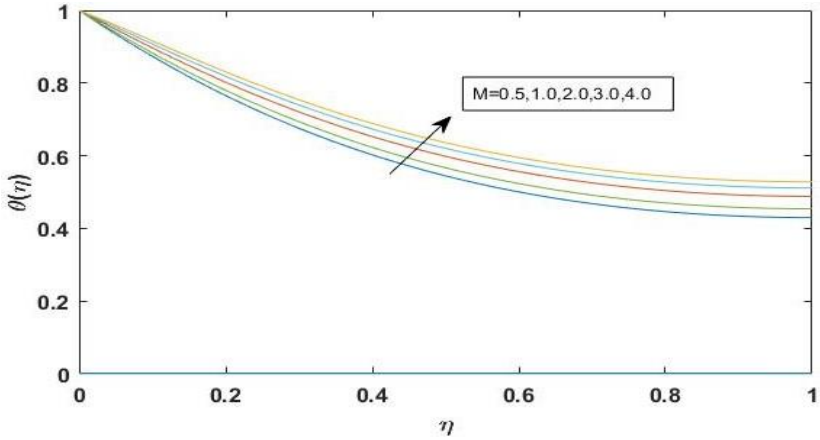

Figure 3. Temperature profiles of fluid flow versus **η** when $c = 0.2, S = 1.3, Ec = 0.5, Pr = 1.0, \zeta = 0.2, B^{*} = 0.5$.

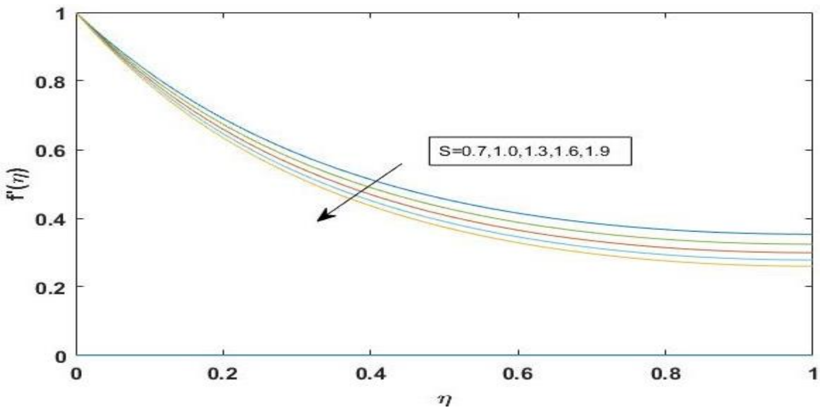

Figure 4. Velocity profiles of fluid flow versus **η** when $c = 0.2, M = 0.5, Ec = 0.5, Pr = 1.0, \zeta = 0.2, B^{*} = 0.5$.

Figure 6 demonstrates the influence of viscosity variation parameter on velocity profile of fluid. It is observed that increasing in viscosity variation parameter decreases the velocity of fluid. [4] It is observed from figure 7 that the temperature of fluid decreases towards inciting values of Prandtl number. Since the Prandtl number is the ratio of momentum and thermal diffusivity, therefore, when Prandtl number increases, the momentum diffusivity dominated thermal diffusivity and hence the temperature of fluid decreases.

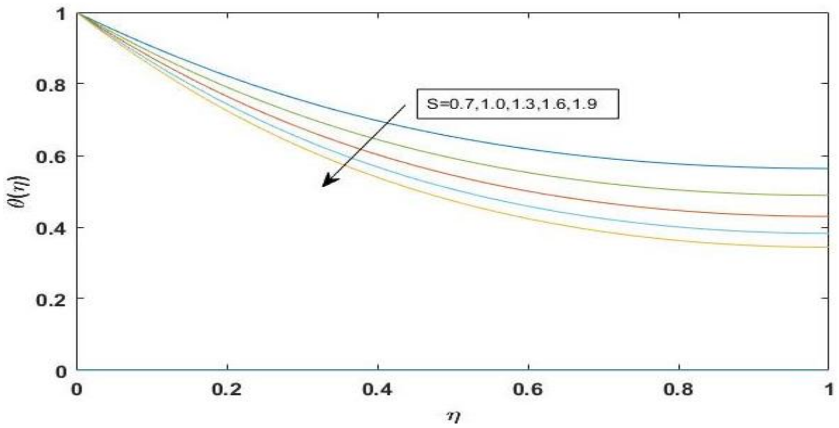

Figure 5. Temperature profiles of fluid flow versus **η** when $c = 0.2, M = 0.5, Ec = 0.5, Pr = 1.0, \zeta = 0.2, B^* = 0.5$.

Figure 8 illustrates that the temperature of fluid rises towards the increasing values of heat source/sink parameter. It is noted from figure 9 that fluid temperature leads an increasing behaviour with the inciting values of Eckert number. Physically Eckert number signifies the conversion of kinetic energy into internal energy. The resistance of fluid particles creates heat therefore the fluid temperature increases. Figure 10 depicts that temperature profile of fluid increases with increasing value of thermal conductivity parameter. The joule heating parameter is distinguished by the product of magnetic parameter and Eckert number. It is obvious from figure 4 and figure 9 that temperature increases with inciting values of joule heating parameter.

The numerical data of skin-friction coefficient and local Nusselt number at the surface of the sheet for various physical parameters are shown in table 1.

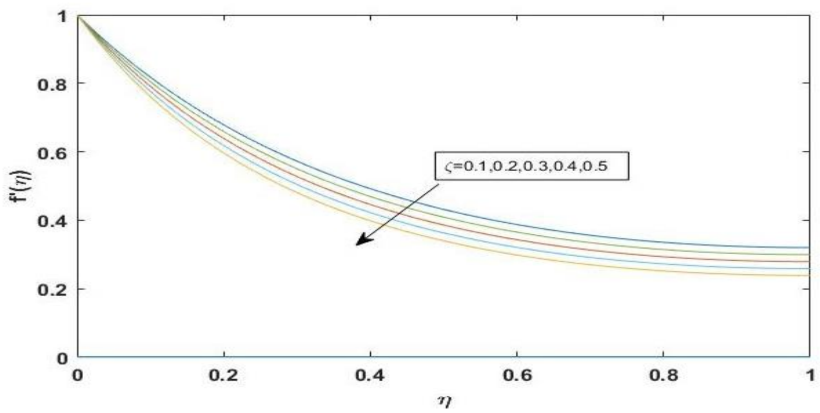

Figure 6. Velocity profiles of fluid flow versus **η** when $c = 0.2, M = 0.5, Ec = 0.5, Pr = 1.0, S = 1.3, B^* = 0.5$.

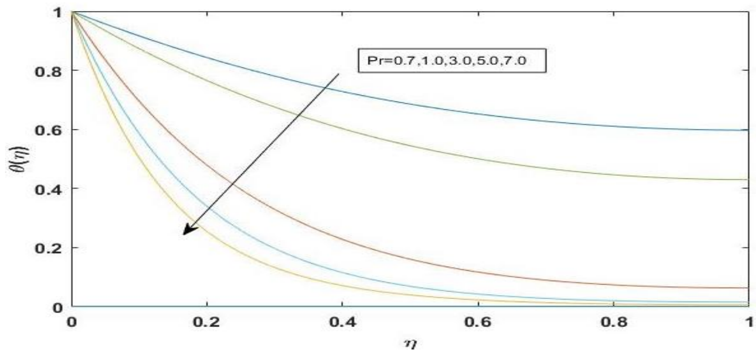

Figure 7. Temperature profiles of fluid flow versus **η** when $c = 0.2, M = 0.5, Ec = 0.5, \zeta = 0.2, S = 1.3, B^* = 0.5$.

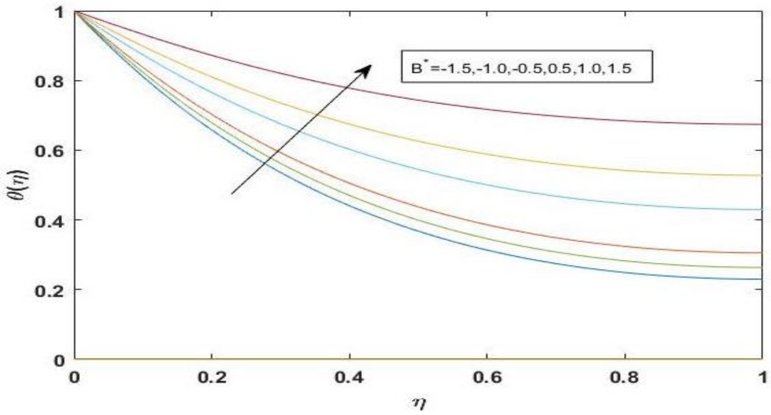

Figure 8. Temperature profiles of fluid flow versus **η** when $c = 0.2, M = 0.5, Ec = 0.5, \zeta = 0.2, S = 1.3, Pr = 1.0$.

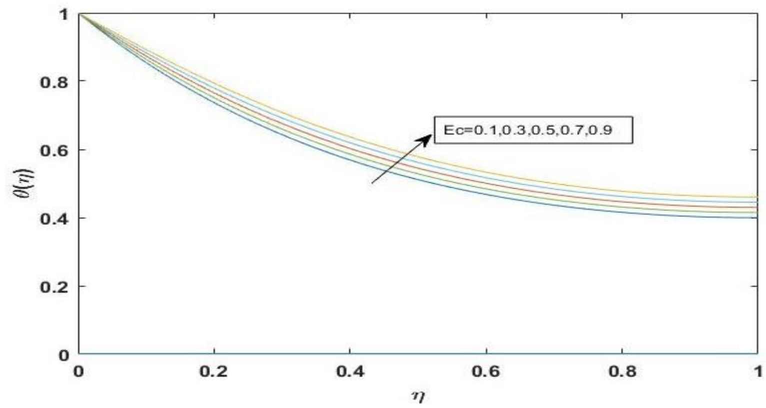

Figure 9. Temperature profiles of fluid flow versus **η** when $c = 0.2, M = 0.5, B^* = 0.5, \zeta = 0.2, S = 1.3, Pr = 1.0$.

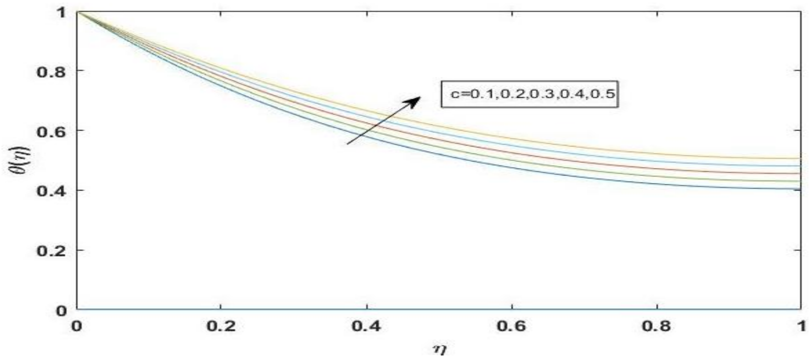

Figure 10. Temperature profiles of fluid flow versus **η**
when $M = 0.5, S = 1.3, Ec = 0.5, Pr = 1.0, \zeta = 0.2, B^{*} = 0.5$.

Table 1. Numerical values of skin friction coefficient and Nusselt number for thin liquid film flow at the surface for several useful concrete variables.

c	M	S	Pr	Ec	ζ	B^{*}	$-f''(0)$	$-\theta'(0)$
0.1	0.5	1.3	1.0	0.5	0.2	0.5	2.2606316	1.4739095
0.2							2.2594992	1.3521775
0.3							2.2584625	1.2461183
0.4							2.2575098	1.1524784
0.5							2.2566301	1.0688769
0.2	0.5	1.3	1.0	0.5	0.2	0.5	2.2594992	1.3521755
	1.0						2.4668418	1.2374282
	2.0						2.8151466	1.0507862
	3.0						3.1052421	0.9017682
	4.0						3.3578112	0.7768258
0.2	0.5	0.7	1.0	0.5	0.2	0.5	2.0132452	1.0113810
		1.0					2.1426583	1.1972818
		1.3					2.2594992	1.3521775
		1.6					2.3662374	1.4850995
		1.9					2.4647406	1.6019401
0.2	0.5	1.3	0.7	0.5	0.2	0.5	2.2542919	0.8770706
			1.0				2.2594992	1.3521775

c	M	S	Pr	Ec	ζ	B^*	$-f''(0)$	$-\theta'(0)$
			3.0				2.2788424	3.5414235
			5.0				2.2886377	5.2073916
			7.0				2.2950919	6.7323237
0.2	0.5	1.3	1.0	0.1	0.2	0.5	2.2614567	1.6119527
				0.3			2.2604772	1.4820971
				0.5			2.2594992	1.3521775
				0.7			2.2585225	1.2221948
				0.9			2.2575470	1.0921498
0.2	0.5	1.3	1.0	0.5	0.1	0.5	2.0834438	1.3695458
					0.2		2.2594992	1.3521775
					0.3		2.4492882	1.3346568
					0.4		2.6538726	1.3170168
					0.5		2.8744189	1.2992927
0.2	0.5	1.3	1.0	0.5	0.2	-1.5	2.2667266	2.0747351
						-1.0	2.2653318	1.9277282
						-0.5	2.2637243	1.7633376
						0.5	2.2594992	1.3521775
						1.0	2.2565092	1.0749657
						1.5	2.2523522	0.7019724

5. Conclusion

In present study, we considered an unsteady laminar thin film flow of viscous incompressible fluid over a permeable stretching sheet. The effects of various physical parameters such as magnetic field parameter, heat source/sink parameter, joule heating parameter, and some variable fluid properties are also investigated. On the basis of above study, the following conclusions are made:

(1) Velocity of fluid flow in thin liquid film decreases with the inciting values of magnetic parameter and temperature of fluid rises with the increasing value of magnetic parameter.

(2) The fluid temperature leads an increasing behavior with the hike of Eckert number or Joule heating parameter or thermal conductivity parameter.

(3) The temperature of thin film flow decreases with inciting values of Prandtl number or unsteadiness parameter.

(4) The local Nusselt number increases with the increase of Prandtl number and Eckert number while it decreases due to increase in magnetic field parameter.

(5) Skin friction coefficient leads an increasing behavior with the increasing value of both magnetic field parameter and unsteadiness parameter.

Nomenclature

a, b	Constants	x, y	Cartesian coordinates
B	Strength of magnetic field	**Greek symbols**	
B^*	Coefficient of heat source/sink	σ	Electrical conductivity
c	Thermal conductivity parameter	ρ	Density of the fluid
c_p	Specific heat at constant pressure	μ	Coefficient of viscosity
C_f	Skin friction coefficient	μ_0	Viscosity of the fluid at the slot
d	Constant	κ	Thermal conductivity of the fluid
f	Dimensionless stream function	κ_0	Conductivity of the fluid at the slot
h	Thickness of thin liquid film	η	Similarity variable
M	Magnetic parameter	υ	Kinematic viscosity
Nu	Nusselt number	τ_s	Shearing stress
$\Pr$	Prandtl number	θ	Dimensionless temperature
q_s	Rate of heat transfer	ζ	Viscosity parameter
Q	Heat source/sink	ζ_0	Positive constant
Re	Reynolds number	Ψ	Stream function
S	Unsteadiness parameter	**Subscripts**	
T	Temperature	0	Condition at slot
T_0	Temperature of fluid at the slot	ref	Reference value
T_{ref}	Constant reference temperature	s	Condition at sheet
T_s	Temperature of stretching sheet	**Superscripts**	
U	Stretching velocity	$'$	First derivative
u	Fluid velocity in x-direction	$''$	Second derivative
v	Fluid velocity in y-direction	$'''$	Third derivative

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