Journal of Rajasthan Academy of Physical Sciences
ISSN: 0972-6306; URL: http://raops.org.in
Vol.21, No.3&4, July-December, 2022, 189-194

ON THE STUDY OF THE EXPONENTIAL DIOPHANTINE EQUATION $u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$

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Abstract: In this paper, we shall discuss the solution of the non-linear Exponential Diophantine equation $u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$, where u is a number of the form $11r + 1$ and $\alpha, \beta, \gamma, v, w, r \in W$, here W is the set of whole numbers. We illustrate that this exponential Diophantine equation has no solution in non-negative integers.

Keywords: Exponential Diophantine equations, Integral solutions.

Mathematics Subject Classification: 11D61, 11D72, 11D45.

1. Introduction

Number theory is an interesting part of mathematics, and the Diophantine equation plays an important role in the theory of numbers. The Theory of Numbers is a wide component of mathematics that is related to many other branches of mathematics.

There are many applications of number theory in computer science. Number theorists use computers for many tasks, including factoring large integers, determining primes, testing conjectures, and solving many other problems.

The purpose of any Diophantine equation is to find solutions for all the unknowns in the equation. When a Diophantine equation contains one or more unknowns, we try to write all the unknowns in terms of one unknown.

In present years, many researchers work on the solution of Diophantine equation of the form $p^x + q^y = z^2$, where p, q are distinct primes, $x, y, z \in W$, and W is the set of whole numbers. In [1], Acu proved that the Diophantine equation $2^x + 5^y = z^2$ has only two solutions $\{x = 3, y = 0, z = 3\}$, and $\{x = 2, y = 1, z = 3\}$. In [2], Burshtein proved that the equation $p^x + (p + 4)^y = z^2$, where $x, y, z \in W$ and $p, p + 4$ are primes with $p > 3$ has no solution.

In [3], Burshtein discussed the Diophantine equation $2^x + p^y = z^2$, where x, y, z are natural numbers. In [4], Gupta and Kumar discussed the Exponential Diophantine equation $n^x + (n + 3m)^y = z^{2k}$, where n is a number of the form $6r + 1$ and

$x, y, z, m, k, r \in W$. In [5], Gupta and Kumar discussed the solution of Exponential Diophantine equation $a^u + (a + 5b)^v = c^{2w}$, where a is a number of the form $5r + 1$ and $u, v, w, b, c, r \in W$. In [6], Kumar et al. discussed the Exponential Diophantine equation $601^p + 619^q = r^2$, where $p, q, r \in W$. In [7], Kumar et al. discussed the equation $p^x + (p + 12)^y = z^2$ and demonstrate that it has no solution in W under certain conditions.

In [8], Kumar et al. discussed the equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$, where x, y, z are natural numbers. In [9], Mishra et al. studied the Diophantine equation $211^\alpha + 229^\beta = \gamma^2$, where $\alpha, \beta, \gamma \in W$. In [10], Sroysang studied the Diophantine equation $31^x + 32^y = z^2$ and show that this equation has no solution in W. In [11], Sroysang proved that the Diophantine equation $2^x + 19^y = z^2$ has unique solution in W, which is $\{x = 3, y = 0, z = 3\}$. In [12], Sroysang proved that the Diophantine equation $8^x + 13^y = z^2$ has unique solution in W, which is $\{x = 1, y = 0, z = 3\}$.

In the present paper, we proved that the Exponential type non-linear Diophantine equation $u^\alpha + (u + 11v)^\beta = w^{2\gamma}$, where u is a number of the form $11r + 1$ and $\alpha, \beta, \gamma, v, w, r \in W$, has no solution in W.

2. Preliminaries

Lemma 2.1. If u is a number of the form $11r + 1$, then the Exponential type equation $u^\alpha + 1 = w^{2\gamma}$, has no solution in W, where $\alpha, \gamma, w, r \in W$.

Proof. Here we consider two cases.

Case 1. If u^α is odd, then $w^{2\gamma}$ is even $\Rightarrow w$ is even

$\Rightarrow w^2 \equiv 0, 1, 3, 4, 5, 9 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 0, 1, 3, 4, 5, 9 (mod\ 11)$

Case 2. If u^α is even, then $w^{2\gamma}$ is odd $\Rightarrow w$ is odd

$\Rightarrow w^2 \equiv 0, 1, 3, 4, 5, 9 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 0, 1, 3, 4, 5, 9 (mod\ 11)$

Now, $u = 11r + 1 \Rightarrow u \equiv 1 (mod\ 11) \Rightarrow u^\alpha \equiv 1 (mod\ 11) \Rightarrow u^\alpha + 1 \equiv 2 (mod\ 11)$

Thus, we obtain $w^{2\gamma} \equiv 2 (mod\ 11)$. This is a contradiction in each case.

Lemma 2.2. If u is a number of the form $11r + 1$, then the Exponential type equation $1 + (u + 11v)^\beta = w^{2\gamma}$, has no solution in W, where $v, w, \beta, \gamma, r \in W$.

Proof. Here we consider two cases.

Case 1. If $(u + 11v)^\beta$ is odd, then $w^{2\gamma}$ is even $\Rightarrow w$ is even

$\Rightarrow w^2 \equiv 0, 1, 3, 4, 5, 9 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 0, 1, 3, 4, 5, 9 (mod\ 11)$

Case 2. If $(u+11v)^{\beta}$ is even, then $w^{2\gamma}$ is odd $\Rightarrow w$ is odd

$\Rightarrow w^{2} \equiv 0,1,3,4,5,9 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 0,1,3,4,5,9 (mod\ 11)$

Now, $u = 11r+1 \Rightarrow u \equiv 1 (mod\ 11) \Rightarrow u+11v \equiv 1 (mod\ 11) \Rightarrow (u+11v)^{\beta} \equiv 1 (mod\ 11)$

$\Rightarrow 1+(u+11v)^{\beta} \equiv 2 (mod\ 11)$

Thus, we obtain $w^{2\gamma} \equiv 2 (mod\ 11)$. This is a contradiction in each case.

3. Main Theorem

Theorem 3.1. If u is a number of the form $11r+1$, then the Exponential type equation $u^{\alpha}+(u+11v)^{\beta} = w^{2\gamma}$, has no solution in W, where $\alpha,\beta,\gamma,v,w,r \in W$.

Proof. Here we consider three cases.

Case I. If $\alpha = 0$, then $1+(u+11v)^{\beta} = w^{2\gamma}$, which has no solution by Lemma 2.2.

Case II. If $\beta = 0$, then $u^{\alpha}+1 = w^{2\gamma}$, which has no solution by Lemma 2.1.

Case III. If $\alpha \geq 1$ and $\beta \geq 1$ then both u^{α} and $(u+11v)^{\beta}$ can be even or odd.

Subcase 1. If u^{α} and $(u+11v)^{\beta}$ both are odd or both even, then $w^{2\gamma}$ is even

$\Rightarrow w$ is even

$\Rightarrow w^{2} \equiv 0,1,3,4,5,9 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 0,1,3,4,5,9 (mod\ 11)$

Subcase 2. If u^{α} is odd and $(u+11v)^{\beta}$ is even or u^{α} is even and $(u+11v)^{\beta}$ is odd, then $w^{2\gamma}$ is odd

$\Rightarrow w$ is odd

$\Rightarrow w^{2} \equiv 0,1,3,4,5,9 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 0,1,3,4,5,9 (mod\ 11)$

Now, $u = 11r+1 \Rightarrow u \equiv 1 (mod\ 11)$ and $u+11v \equiv 1 (mod\ 11)$

$\Rightarrow u^{\alpha} \equiv 1 (mod\ 11)$ and $(u+11v)^{\beta} \equiv 1 (mod\ 11)$

$\Rightarrow u^{\alpha}+(u+11v)^{\beta} \equiv 2 (mod\ 11)$

$\Rightarrow w^{2\gamma} \equiv 2 (mod\ 11)$, this is a contradiction in each subcase.

Hence, the Exponential Diophantine equation $u^{\alpha}+(u+11v)^{\beta} = w^{2\gamma}$ has no solution in W.

Corollary 3.1.1. If u is a number of the form $11r+1$, then the Exponential type equation $u^{\alpha}+(u+11v)^{\beta} = k^{4\gamma}$ has no solution in W, where $\alpha,\beta,\gamma,v,k,r \in W$.

Proof. Let $k^2 = w$, then this equation becomes $u^\alpha + (u + 11v)^\beta = w^{2\gamma}$, which has no solution by Theorem 3.1.

Corollary 3.1.2. If u is a number of the form $11r + 1$, then the Exponential type equation $u^\alpha + (u + 11v)^\beta = t^{2\gamma(\gamma+1)}$ has no solution in W, where $\alpha, \beta, \gamma, v, t, r \in W$.

Proof. Let $t^{(\gamma+1)} = w$, then this equation becomes $u^\alpha + (u + 11v)^\beta = w^{2\gamma}$, which has no solution by Theorem 3.1.

Corollary 3.1.3. If u is a number of the form $11r + 1$, then the Exponential type equation $u^\alpha + (u + 22k)^\beta = w^{2\gamma}$ has no solution in W, where $\alpha, \beta, \gamma, k, r, w \in W$.

Proof. Let $2k = v$, then this equation becomes $u^\alpha + (u + 11v)^\beta = w^{2\gamma}$, which has no solution by Theorem 3.1.

4. Some Special Cases

Problem 4.1. The non-linear Diophantine equation $12^\alpha + 78^\beta = w^2, \alpha, \beta, w \in W$ has no solution in non-negative integers.

Solution: Here $u = 12$, which is of the form $11r + 1$ for $r = 1$ and $(u + 11v) = 78$ i.e., $v = 6$, so there is no solution to the equation $12^\alpha + 78^\beta = w^2$, according to theorem 3.1.

Problem 4.2. The non-linear Diophantine equation $166^a + 177^b = c^2, a, b, c \in W$ has no solution in non-negative integers.

Solution: Here $u = 166$, which is of the form $11r + 1$ for $r = 15$ and $(u + 11v) = 177$ i.e., $v = 1$, so there is no solution to the equation $166^a + 177^b = c^2$, according to theorem 3.1.

Note: In this paper, we have shown that the Exponential Diophantine equation $u^\alpha + (u + 11v)^\beta = w^{2\gamma}$, where u is a number of the form $11r + 1$ and $\alpha, \beta, \gamma, v, w, r \in W$, has no solution in W. If we change the value of v, r, and γ, we get an infinite number of Exponential Diophantine equations with no solution in W. The tables below contains some Exponential Diophantine equations with no solution for specific values of v, r, and γ.

Table 1

Value of r	**Value of v**	**Value of γ**	$u^\alpha + (u + 11v)^\beta = w^{2\gamma}$	**Solution of the equation**
0	1	1	$1^\alpha + 12^\beta = w^2$	No Solution
1	1	1	$12^\alpha + 23^\beta = w^2$	No Solution
2	1	1	$23^\alpha + 34^\beta = w^2$	No Solution
3	1	1	$34^\alpha + 45^\beta = w^2$	No Solution
4	1	1	$45^\alpha + 56^\beta = w^2$	No Solution

Table 2

Value of r	Value of v	Value of γ	$u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$	Solution of the equation
0	2	1	$1^{\alpha} + 23^{\beta} = w^2$	No Solution
1	2	1	$12^{\alpha} + 34^{\beta} = w^2$	No Solution
2	2	1	$23^{\alpha} + 45^{\beta} = w^2$	No Solution
3	2	1	$34^{\alpha} + 56^{\beta} = w^2$	No Solution
4	2	1	$45^{\alpha} + 67^{\beta} = w^2$	No Solution

Table 3

Value of r	Value of v	Value of γ	$u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$	Solution of the equation
0	3	1	$1^{\alpha} + 34^{\beta} = w^2$	No Solution
1	3	1	$12^{\alpha} + 45^{\beta} = w^2$	No Solution
2	3	1	$23^{\alpha} + 56^{\beta} = w^2$	No Solution
3	3	1	$34^{\alpha} + 67^{\beta} = w^2$	No Solution
4	3	1	$45^{\alpha} + 78^{\beta} = w^2$	No Solution

Table 4

Value of r	Value of v	Value of γ	$u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$	Solution of the equation
0	4	1	$1^{\alpha} + 45^{\beta} = w^2$	No Solution
1	4	1	$12^{\alpha} + 56^{\beta} = w^2$	No Solution
2	4	1	$23^{\alpha} + 67^{\beta} = w^2$	No Solution
3	4	1	$34^{\alpha} + 78^{\beta} = w^2$	No Solution
4	4	1	$45^{\alpha} + 89^{\beta} = w^2$	No Solution

Table 5

Value of r	Value of v	Value of γ	$u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$	Solution of the equation
0	5	1	$1^{\alpha} + 56^{\beta} = w^2$	No Solution
1	5	1	$12^{\alpha} + 67^{\beta} = w^2$	No Solution
2	5	1	$23^{\alpha} + 78^{\beta} = w^2$	No Solution
3	5	1	$34^{\alpha} + 89^{\beta} = w^2$	No Solution
4	5	1	$45^{\alpha} + 100^{\beta} = w^2$	No Solution

5. Conclusion

In this paper, we prove that the Exponential Diophantine equation $u^{\alpha} + (u + 11v)^{\beta} = w^{2\gamma}$, where u is a number of the form $11r + 1$ and $\alpha, \beta, \gamma, v, w, r \in W$ has no solution in W.

Acknowledgement: The authors are thankful to the Referee for valuable comments and suggestions.

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