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PROPERTIES OF GENERALIZED FIBONACCI AND LUCAS POLYNOMIALS

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Abstract: The renowned Fibonacci and Lucas polynomials possess various astounding properties and identities. The Fibonacci polynomial has been generalized in many ways by conserving the recurrence relation and others by preserving the initial conditions. In this paper, we have defined generalized Fibonacci and Lucas polynomial and with the help of generating function and Binet's formula, we proved famous identities for the same in our settings.

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1. Introduction

To give the solution to the famous rabbit problem i.e. “How many pairs of rabbits are born of one pair in a year?” The Italian Mathematician Leonardo Pisano in 1180 discovered the Fibonacci sequence. We can use Fibonacci numbers in various fields of life. It can be found also in beautiful flowers, organs of human body and in animal growth.

Fibonacci polynomial is defined as a polynomial sequence which can be considered as a generalization of the Fibonacci numbers. In similar way, polynomials generated from Lucas numbers are called as Lucas polynomials. Fibonacci and Lucas polynomials are extensively explored, appear in different framework and possess various outstanding properties.

The Fibonacci polynomial studied by Catalan is defined by the recurrence relation

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x);\ n \geq 2 \text{ with } f_0(x) = 0,\ f_1(x) = 1.$$

The Lucas polynomial is defined in [7] by the recurrence relation

$L_n(x) = xL_{n-1}(x) + L_{n-2}(x)\,;\, n \geq 2$ where $L_0(x) = 2,\, L_1(x) = x$.

Pell polynomials defined in [6] is given by

$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x)\,;\, n \geq 2$ with $P_0(x) = 0, P_1(x) = 1$

The generating function for Fibonacci and Lucas polynomial defined in [3] is given by

$$\sum_{n=0}^{\infty} f_n(x)t^n = t(1-xt-t^2)^{-1}\ ,\ \sum_{n=0}^{\infty} L_n(x)t^n = (2-xt)(1-xt-t^2)^{-1}.$$

In this Paper, we define generalized Fibonacci polynomial b

$$h_n(x) = p_1(x)h_{n-1}(x) + p_2(x)h_{n-2}(x)\,;\, n \geq 2 \text{ with } h_0(x) = a+b, h_1(x) = 2a+1 \tag{1}$$

where, a and b are integers, $p_1(x)$ and $p_2(x)$ are polynomials in x and the generalized Lucas polynomial is given by

$t_n(x) = q_1(x)t_{n-1}(x) + q_2(x)t_{n-2}(x)\,;\, n \geq 2$ where $t_0(x) = a,\, t_1(x) = q_1(x)$.

Where, a is an integer, $q_1(x)$ and $q_2(x)$ are polynomials in x.

For $p_1(x) = p_2(x) = 1, a = 0, b = 0$, we obtain classical Fibonacci sequence and $q_1(x) = q_2(x) = 1, a = 2$, we obtain classical Lucas sequence.

Our generalized Fibonacci polynomial possess generating function which is proved in Lemma 1.1 and is given by

$$\sum_{n=0}^{\infty} h_n(x)t^n = \frac{[(a+b)(1-p_1(x)t)+(2a+1)t]}{(1-p_1(x)t-p_2(x)t^2)}.$$

Similarly, the generating function of our generalized Lucas polynomial which can be proved in similar manner as discussed in lemma 1.1 is given by

$$\sum_{n=0}^{\infty} t_n(x)t^n = \frac{[a+(1-a)\,\mathrm{q}_1(x)t]}{(1-q_1(x)t-q_2(x)t^2)}.$$

Easy generalization of the calculations [6], we have Binet's Formula for generalized Fibonacci and Lucas polynomial defined in equation (1) and (2) is respectively given by

$h_n\left(x\right)=\left(A\alpha^n+B\beta^n\right)$ where $A=\frac{\left(2a+1\right)-\left(a+b\right)\beta}{\alpha-\beta}, B=\frac{\left(a+b\right)\alpha-\left(2a+1\right)}{\alpha-\beta}$ and $t_n\left(x\right)=K\left(\alpha^n+\beta^n\right)$ where $K=\frac{a}{2}$.

This paper involves the derivation of famous identities such as Catalan's, Cassini's, d'Ocagne's and many more identities for both the generalized Fibonacci and generalized Lucas polynomial with the help of generating function and Binet's Formula respectively. In recent years, many new identities have been derived for a generalized Fibonacci and generalized Lucas polynomial and some of them can be studied in [1, 4].

Lemma 1.1: The generating function for a generalized Fibonacci polynomial defined in equation (1) is given by

$$\sum_{n=0}^{\infty}h_n(x)t^n=\frac{[(a+b)(1-p_1(x)t)+(2a+1)t]}{(1-p_1(x)t-p_2(x)t^2)}.$$

Proof: Put $n=n+1$ in equation (1), we have

$$h_{n+1}\left(x\right)=p_1\left(x\right)h_n\left(x\right)+p_2\left(x\right)h_{n-1}\left(x\right);\ n\geq 3 \tag{2}$$

Let $F(t)=\sum_{n=0}^{\infty}h_n(x)t^n$ (3)

From equation (2) we have,

$$\sum_{n\geq 1}h_{n+1}(x)t^n=p_1(x)\sum_{n\geq 1}h_n(x)t^n+p_2(x)\sum_{n\geq 1}h_{n-1}(x)t^n \tag{4}$$

Now,

$$\sum_{n\geq 1}h_n(x)t^n=\sum_{n\geq 1}h_n(x)t^n+h_0(x)-h_0(x)=F(t)-(a+b) \tag{5}$$

and

$$\sum_{n\geq 1}h_{n-1}(x)t^n=tF(t) \tag{6}$$

Therefore, R.H.S of equation (4) becomes

$$\sum_{n\geq 1}h_{n+1}(x)t^n=p_1\left(x\right)[F(t)-(a+b)]+p_2\left(x\right)tF(t) \tag{7}$$

Now,

$$\sum_{n\geq 1} h_{n+1}(x)t^{n} = \frac{1}{t}\left[F(t)-(a+b)-t(2a+1)\right]$$

Therefore, equation (7) becomes

$$\frac{1}{t}\left[F(t)-(a+b)-t(2a+1)\right]= p_1\left(x\right)[F(t)-(a+b)]+ p_2\left(x\right)tF(t)$$

$$\sum_{n=0}^{\infty} h_n(x)t^{n} = \frac{[(a+b)(1-p_1\left(x\right)t)+(2a+1)t]}{(1-p_1\left(x\right)t-p_2\left(x\right)t^{2})}.$$

2. Some Identities of Generalized Fibonacci Polynomial

In this section, we investigate some of the identities of our generalized Fibonacci polynomial with the help of a generating function and Binet's formula.

Theorem 2.1

If the n$^{\text{th}}$ term of a generalized Fibonacci polynomial is $h_n(x)$, then

$$h_n^{'}(x)-p_1\left(x\right)h_{n-1}^{'}(x)-p_2\left(x\right)h_{n-2}^{'}(x)=p_1^{'}\left(x\right)h_{n-1}(x)+p_2^{'}\left(x\right)h_{n-2}(x),\ n\geq 2$$

Proof: We know that the generating function of generalized Fibonacci polynomial is given by

$$\sum_{n=0}^{\infty} h_n(x)t^{n} = [(a+b)(1-p_1\left(x\right)t)+(2a+1)t](1-p_1\left(x\right)t-p_2\left(x\right)t^{2})^{-1}$$

Differentiating both sides with respect to x

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{'}(x)t^{n} &= (-1)\left(1-p_1\left(x\right)t-p_2\left(x\right)t^{2}\right)^{-2}\left(-p_1^{'}\left(x\right)t-p_2^{'}\left(x\right)t^{2}\right)\left[(a+b)\left(1-p_1\left(x\right)t\right)+(2a+1)t\right]\\ &\quad +(a+b)\left(-p_1^{'}\left(x\right)t\right)\left(1-p_1\left(x\right)t-p_2\left(x\right)t^{2}\right)^{-1}\end{aligned}$$

$$\begin{aligned}\left(1-p_1\left(x\right)t-p_2\left(x\right)t^{2}\right)\sum_{n=0}^{\infty} h_n^{'}(x)t^{n} &= \left[(a+b)(-p_1^{'}\left(x\right)t)\right]\\ &\quad +\left(1-p_1\left(x\right)t-p_2\left(x\right)t^{2}\right)^{-1}\left(p_1^{'}\left(x\right)t+p_2^{'}\left(x\right)t^{2}\right)\left[(a+b)\left(1-p_1\left(x\right)t\right)+(2a+1)t\right]\end{aligned}$$

$$\left(1-p_1\left(x\right)t-p_2\left(x\right)t^{2}\right)\sum_{n=0}^{\infty} h_n^{'}(x)t^{n} = \left[(a+b)(-p_1^{'}\left(x\right)t)\right]+\sum_{n=0}^{\infty} h_n(x)t^{n}\left(p_1^{'}\left(x\right)t+p_2^{'}\left(x\right)t^{2}\right)$$

$$\begin{aligned}\sum_{n=0}^{\infty} h_n^{'}(x)t^{n}-p_1\left(x\right)\sum_{n=0}^{\infty} h_n^{'}(x)t^{n+1}-p_2\left(x\right)\sum_{n=0}^{\infty} h_n^{'}(x)t^{n+2} &= p_1^{'}\left(x\right)\sum_{n=0}^{\infty} h_n(x)t^{n+1}+p_2^{'}\left(x\right)\sum_{n=0}^{\infty} h_n(x)t^{n+2}\\ &\quad -(a+b)(p_1^{'}\left(x\right)t)\end{aligned}$$

Equating the coefficient of t^n on both sides, we have

$$h_n^{'}(x) - p_1(x)h_{n-1}^{'}(x) - p_2(x)h_{n-2}^{'}(x) = p_1^{'}(x)h_{n-1}(x) + p_2^{'}(x)h_{n-2}(x)$$

Theorem 2.2

Let $h_n(x)$ be the nth term of a generalized Fibonacci polynomial, then

$$p_1^{'}(x)nh_n(x) - (n-1)\left\{p_1(x)p_1^{'}(x) - p_2^{'}(x)\right\}h_{n-1}(x) - (n-2)\left\{p_2(x)p_1^{'}(x) + p_2^{'}(x)p_1(x)\right\}h_{n-2}(x)$$
$$-p_2^{'}(x)p_2(x)(n-3)h_{n-3}(x) = p_1(x)h_n^{'}(x) - \left\{p_1^2(x) + 2p_2(x)\right\}h_{n-1}^{'}(x) - 3p_1(x)p_2(x)h_{n-2}^{'}(x)$$
$$-2p_2^2(x)h_{n-3}^{'}(x)\ ;\ n \geq 3$$

Proof: We know that the generating function of generalized Fibonacci polynomial is given by

$$\sum_{n=0}^{\infty} h_n(x)t^n = [(a+b)(1-p_1(x)t) + (2a+1)t](1-p_1(x)t - p_2(x)t^2)^{-1}$$

Differentiate both sides with respect to t

$$\sum_{n=0}^{\infty} nh_n(x)t^{n-1} = \left[-p_1(x)(a+b) + (2a+1)\right]\left(1-p_1(x)t-p_2(x)t^2\right)^{-1}$$
$$+(-1)\left(1-p_1(x)t-p_2(x)t^2\right)^{-2}\left(-p_1(x) - 2p_2(x)t\right)\left[(a+b)\left(1-p_1(x)t\right) + (2a+1)t\right]$$

---(I)

Differentiating both sides with respect to x

$$\sum_{n=0}^{\infty} h_n^{'}(x)t^n = (-1)\left(1-p_1(x)t-p_2(x)t^2\right)^{-2}\left(-p_1^{'}(x)t - p_2^{'}(x)t^2\right)\left[(a+b)\left(1-p_1(x)t\right)+(2a+1)t\right]$$
$$+(a+b)\left(-p_1^{'}(x)t\right)\left(1-p_1(x)t-p_2(x)t^2\right)^{-1}$$

On dividing both sides by t

$$\sum_{n=0}^{\infty} h_n^{'}(x)t^{n-1} = \left(1-p_1(x)t-p_2(x)t^2\right)^{-2}\left(p_1^{'}(x)+p_2^{'}(x)t\right)\left[(a+b)\left(1-p_1(x)t\right)+(2a+1)t\right]$$
$$-(a+b)\left(p_1^{'}(x)\right)\left(1-p_1(x)t-p_2(x)t^2\right)^{-1}$$

$$\sum_{n=0}^{\infty} h_n^{'}(x)t^{n-1} + (a+b)\left(p_1^{'}(x)\right)\left(1-p_1(x)t-p_2(x)t^2\right)^{-1}$$
$$= \left(1-p_1(x)t-p_2(x)t^2\right)^{-2}\left(p_1^{'}(x)+p_2^{'}(x)t\right)\left[(a+b)\left(1-p_1(x)t\right)+(2a+1)t\right]$$

---(II)

On substituting the value of R.H.S of equation (II) in equation (I), We have

$$\sum_{n=0}^{\infty} nh_n(x)t^{n-1} = \left[-p_1(x)(a+b)+(2a+1)\right]\left(1-p_1(x)t-p_2(x)t^2\right)^{-1}$$
$$+\frac{\left(p_1(x)+2p_2(x)t\right)}{\left(p_1'(x)+p_2'(x)t\right)}\left\{\sum_{n=0}^{\infty} h_n'(x)t^{n-1}+(a+b)\left(p_1'(x)\right)\left(1-p_1(x)t-p_2(x)t^2\right)^{-1}\right\}$$

On multiplying both sides by $\left(p_1'(x)+p_2'(x)t\right)\left(1-p_1(x)t-p_2(x)t^2\right)$ we have,

$$\left(p_1'(x)+p_2'(x)t\right)\left(1-p_1(x)t-p_2(x)t^2\right)\sum_{n=0}^{\infty} nh_n(x)t^{n-1} = \left(p_1'(x)+p_2'(x)t\right)\left[-p_1(x)(a+b)+(2a+1)\right]$$
$$+\left(p_1(x)+2p_2(x)t\right)\left(1-p_1(x)t-p_2(x)t^2\right)\sum_{n=0}^{\infty} h_n'(x)t^{n-1}+\left(p_1(x)+2p_2(x)t\right)(a+b)\left(p_1'(x)\right)$$
$$p_1'(x)\sum_{n=0}^{\infty} nh_n(x)t^{n-1}-p_1(x)p_1'(x)\sum_{n=0}^{\infty} nh_n(x)t^{n}-p_2(x)p_1'(x)\sum_{n=0}^{\infty} nh_n(x)t^{n+1}$$
$$+p_2'(x)\sum_{n=0}^{\infty} nh_n(x)t^{n}-p_2'(x)p_1(x)\sum_{n=0}^{\infty} nh_n(x)t^{n+1}-p_2'(x)p_2(x)\sum_{n=0}^{\infty} nh_n(x)t^{n+2}$$
$$=p_1(x)\sum_{n=0}^{\infty} h_n'(x)t^{n-1}-p_1^2(x)\sum_{n=0}^{\infty} h_n'(x)t^{n}-p_1(x)p_2(x)\sum_{n=0}^{\infty} h_n'(x)t^{n+1}+2p_2(x)\sum_{n=0}^{\infty} h_n'(x)t^{n}$$
$$-2p_1(x)p_2(x)\sum_{n=0}^{\infty} h_n'(x)t^{n+1}-2p_2^2(x)\sum_{n=0}^{\infty} h_n'(x)t^{n+2}(a+b)p_1(x)p_1'(x)+2(a+b)p_2(x)p_1'(x)t$$
$$-(a+b)p_1(x)p_1'(x)-(a+b)p_1(x)p_2'(x)t+p_1'(x)(2a+1)+p_2'(x)(2a+1)t$$

On equating the coefficient of t^{n-1} on both sides, we have

$$p_1'(x)nh_n(x)-(n-1)p_1(x)p_1'(x)h_{n-1}(x)-p_2(x)p_1'(x)(n-2)h_{n-2}(x)$$
$$+p_2'(x)(n-1)h_{n-1}(x)-p_2'(x)p_1(x)(n-2)h_{n-2}(x)-p_2'(x)p_2(x)(n-3)h_{n-3}(x)$$
$$=p_1(x)h_n'(x)-p_1^2(x)h_{n-1}'(x)-p_1(x)p_2(x)h_{n-2}'(x)+2p_2(x)h_{n-1}'(x)$$
$$-2p_1(x)p_2(x)h_{n-2}'(x)-2p_2^2(x)h_{n-3}'(x)$$

$$p_1'(x)nh_n(x)-(n-1)\left\{p_1(x)p_1'(x)-p_2'(x)\right\}h_{n-1}(x)-(n-2)\left\{p_2(x)p_1'(x)+p_2'(x)p_1(x)\right\}h_{n-2}(x)$$
$$-p_2'(x)p_2(x)(n-3)h_{n-3}(x)=p_1(x)h_n'(x)-\left\{p_1^2(x)+2p_2(x)\right\}h_{n-1}'(x)-3p_1(x)p_2(x)h_{n-2}'(x)$$
$$-2p_2^2(x)h_{n-3}'(x).$$

Theorem 2.3

For the generalized Fibonacci polynomial $h_n(x)$, we have (*i*) $h'_{n+1}(x)-p_2(x)h'_{n-1}(x)-p_1(x)h'_n(x)=p'_1(x)h_n(x)+p'_2(x)h_{n-1}(x)$

(*ii*) $h'_{n+1}(x)+\left\{-p_1^2(x)+p_2(x)\right\}h'_{n-1}(x)=(n+1)p'_1(x)h_n(x)+\left\{-(n-1)p_1(x)p'_1(x)+np'_2(x)\right\}h_{n-1}(x)$
$-(n-2)\left\{p_2(x)p'_1(x)-p'_2(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p'_2(x)p_2(x)h_{n-3}(x)$
$+2p_2^2(x)h'_{n-3}(x)+3p_1(x)p_2(x)h'_{n-2}(x)$

(*iii*) $\left\{2p_2(x)-p_1^2(x)\right\}h'_{n-1}(x)=p_1(x)h'_n(x)+np'_1(x)h_n(x)+(n-1)\left\{-p_1(x)p'_1(x)+p'_2(x)\right\}h_{n-1}(x)$
$-(n-2)\left\{p_2(x)p'_1(x)+p'_2(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p'_2(x)p_2(x)h_{n-3}(x)$
$+2p_2^2(x)h'_{n-3}(x)+3p_1(x)p_2(x)h'_{n-2}(x)$

(*iv*) $\left\{2p_2(x)-p_1^2(x)\right\}h'_{n+1}(x)=p_1(x)\left\{p_2(x)-p_1^2(x)\right\}h'_n(x)+p'_1(x)h_n(x)\left\{(n+2)p_2(x)-p_1^2(x)\right\}$
$+(n-1)\left\{-p_1(x)p_2(x)p'_1(x)+np_2(x)p'_2(x)+\left(p_2(x)-p_1^2(x)\right)p'_2(x)\right\}h_{n-1}(x)$
$-(n-2)\left\{p_2^2(x)p'_1(x)+p'_2(x)p_2(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p'_2(x)p_2^2(x)h_{n-3}(x)$
$+2p_2^3(x)h'_{n-3}(x)+3p_1(x)p_2^2(x)h'_{n-2}(x)$

Proof: From equation (1)

$$h_{n+1}(x)-p_2(x)h_{n-1}(x)=p_1(x)h_n(x)$$

Differentiate both sides with respect to x

$$h'_{n+1}(x)-p_2(x)h'_{n-1}(x)-p'_2(x)h_{n-1}(x)=p_1(x)h'_n(x)+p'_1(x)h_n(x)$$

$$h'_{n+1}(x)-p_2(x)h'_{n-1}(x)-p_1(x)h'_n(x)=p'_1(x)h_n(x)+p'_2(x)h_{n-1}(x) \tag{I}$$

Use theorem (2.2) in equation (I), we have

$$h'_{n+1}(x)-p_2(x)h'_{n-1}(x)-\left\{\begin{array}{l} p'_1(x)nh_n(x)-(n-1)\left\{p_1(x)p'_1(x)-p'_2(x)\right\}h_{n-1}(x)- \\ (n-2)\left\{p_2(x)p'_1(x)+p'_2(x)p_1(x)\right\}h_{n-2}(x)-p'_2(x)p_2(x)(n-3)h_{n-3}(x) \\ -\left\{-p_1^2(x)+2p_2(x)\right\}h'_{n-1}(x)+3p_1(x)p_2(x)h'_{n-2}(x)+2p_2^2(x)h'_{n-3}(x) \end{array}\right\}$$
$$=p'_1(x)h_n(x)-p'_2(x)h_{n-1}(x)$$

$$\begin{aligned} &h'_{n+1}(x)+\left\{-p_1^2(x)+p_2(x)\right\}h'_{n-1}(x)=(n+1)p'_1(x)h_n(x)+\left\{-(n-1)p_1(x)p'_1(x)+np'_2(x)\right\}h_{n-1}(x) \\ &-(n-2)\left\{p_2(x)p'_1(x)-p'_2(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p'_2(x)p_2(x)h_{n-3}(x)+2p_2^2(x)h'_{n-3}(x) \\ &+3p_1(x)p_2(x)h'_{n-2}(x) \end{aligned} \tag{II}$$

On subtracting equation (I) from (II)

$$\begin{aligned}&\left\{2p_2(x)-p_1^2(x)\right\}h_{n-1}^{'}(x)=p_1(x)h_n^{'}(x)+np_1^{'}(x)h_n(x)+(n-1)\left\{-p_1(x)p_1^{'}(x)+p_2^{'}(x)\right\}h_{n-1}(x)\\&-(n-2)\left\{p_2(x)p_1^{'}(x)+p_2^{'}(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p_2^{'}(x)p_2(x)h_{n-3}(x)\\&+2p_2^2(x)h_{n-3}^{'}(x)+3p_1(x)p_2(x)h_{n-2}^{'}(x)\end{aligned} \tag{III}$$

On multiplying equation (I) by $\left(p_2(x)-p_1^2(x)\right)$ and (II) by $\left(p_2(x)\right)$ on adding both the equation

$$\begin{aligned}&\left\{2p_2(x)-p_1^2(x)\right\}h_{n+1}^{'}(x)=p_1(x)\left\{p_2(x)-p_1^2(x)\right\}h_n^{'}(x)+p_1^{'}(x)h_n(x)\left\{(n+2)p_2(x)-p_1^2(x)\right\}\\&+(n-1)\left\{-p_1(x)p_2(x)p_1^{'}(x)+np_2(x)p_2^{'}(x)+\left(p_2(x)-p_1^2(x)\right)p_2^{'}(x)\right\}h_{n-1}(x)\\&-(n-2)\left\{p_2^2(x)p_1^{'}(x)+p_2^{'}(x)p_2(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p_2^{'}(x)p_2^2(x)h_{n-3}(x)\\&+2p_2^3(x)h_{n-3}^{'}(x)+3p_1(x)p_2^2(x)h_{n-2}^{'}(x)\end{aligned} \tag{IV}$$

Theorem 2.4

Let $h_n(x)$ be the n$^{\text{th}}$ term of generalized Fibonacci polynomial, then

$$\begin{aligned}&nh_{n+1}^{'}(x)-\left\{(n+2)p_2(x)-p_1^2(x)\right\}h_{n-1}^{'}(x)-(n+1)p_1(x)h_n^{'}(x)+3p_1(x)p_2(x)h_{n-2}^{'}(x)+2p_2^2(x)h_{n-3}^{'}(x)\\&=\left\{p_2^{'}(x)+(n-1)p_1(x)p_1^{'}(x)\right\}h_{n-1}(x)-(n-2)\left\{p_2(x)p_1^{'}(x)-p_2^{'}(x)p_1(x)\right\}h_{n-2}(x)\\&-(n-3)p_2^{'}(x)p_2(x)h_{n-3}(x)\end{aligned}$$

Proof: From theorem (2.3(i)), we have

$$h_{n+1}^{'}(x)-p_2(x)h_{n-1}^{'}(x)-p_1(x)h_n^{'}(x)=p_1^{'}(x)h_n(x)+p_2^{'}(x)h_{n-1}(x) \tag{I}$$

From theorem (2.3(ii))

$$\begin{aligned}&h_{n+1}^{'}(x)+\left\{-p_1^2(x)+p_2(x)\right\}h_{n-1}^{'}(x)=(n+1)p_1^{'}(x)h_n(x)+\left\{-(n-1)p_1(x)p_1^{'}(x)+np_2^{'}(x)\right\}h_{n-1}(x)\\&-(n-2)\left\{p_2(x)p_1^{'}(x)-p_2^{'}(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p_2^{'}(x)p_2(x)h_{n-3}(x)+2p_2^2(x)h_{n-3}^{'}(x)\\&+3p_1(x)p_2(x)h_{n-2}^{'}(x)\end{aligned} \tag{II}$$

On substituting the value of $p_1^{'}(x)h_n(x)$ from equation (I) in equation (II), we have

$$\begin{aligned}&h_{n+1}^{'}(x)+\left\{p_2(x)-p_1^2(x)\right\}h_{n-1}^{'}(x)=(n+1)\left\{h_{n+1}^{'}(x)-p_2(x)h_{n-1}^{'}(x)-p_1(x)h_n^{'}(x)-p_2^{'}(x)h_{n-1}(x)\right\}\\&+\left\{np_2^{'}(x)-(n-1)p_1(x)p_1^{'}(x)\right\}h_{n-1}(x)\\&-(n-2)\left\{p_2(x)p_1^{'}(x)-p_2^{'}(x)p_1(x)\right\}h_{n-2}(x)-(n-3)p_2^{'}(x)p_2(x)h_{n-3}(x)+2p_2^2(x)h_{n-3}^{'}(x)\\&+3p_1(x)p_2(x)h_{n-2}^{'}(x)\end{aligned}$$

$$nh'_{n+1}(x)-\left\{(n+2)p_2(x)-p_1^2(x)\right\}h'_{n-1}(x)-(n+1)p_1(x)h'_n(x)+3p_1(x)p_2(x)h'_{n-2}(x)+2p_2^2(x)h'_{n-3}(x)$$
$$=\left\{p'_2(x)+(n-1)p_1(x)p'_1(x)\right\}h_{n-1}(x)-(n-2)\left\{p_2(x)p'_1(x)-p'_2(x)p_1(x)\right\}h_{n-2}(x)$$
$$-(n-3)p'_2(x)p_2(x)h_{n-3}(x)$$

$$nh'_{n+1}(x)-\left\{(n+2)p_2(x)-p_1^2(x)\right\}h'_{n-1}(x)-(n+1)p_1(x)h'_n(x)+3p_1(x)p_2(x)h'_{n-2}(x)+2p_2^2(x)h'_{n-3}(x)$$
$$=\left\{p'_2(x)+(n-1)p_1(x)p'_1(x)\right\}h_{n-1}(x)-(n-2)\left\{p_2(x)p'_1(x)-p'_2(x)p_1(x)\right\}h_{n-2}(x)$$
$$-(n-3)p'_2(x)p_2(x)h_{n-3}(x)$$

Theorem 2.5

Explicit Sum formula: - For generalized Fibonacci polynomial

$$h_n(x)=(a+b)\left\{\sum_{k=0}^{[n/2]}\binom{n-k}{k}\left(p_1(x)\right)^{n-2k}-\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}\left(p_1(x)\right)^{n-2k-1}\right\}$$
$$+(2a+1)\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}\left(p_1(x)\right)^{n-2k-1}$$

Proof: We know that the generating function of generalized Fibonacci polynomial is given by

$$\sum_{n=0}^{\infty}h_n(x)t^n=[(a+b)(1-p_1(x)t)+(2a+1)t](1-p_1(x)t-p_2(x)t^2)^{-1}$$

$$=[(a+b)(1-p_1(x)t)+(2a+1)t]\sum_{n=0}^{\infty}(p_1(x)+p_2(x)t)^n t^n$$

$$=[(a+b)(1-p_1(x)t)+(2a+1)t]\sum_{n=0}^{\infty}t^n\sum_{k=0}^{n}\binom{n}{k}(p_1(x))^{n-k}t^k$$

$$=[(a+b)(1-p_1(x)t)+(2a+1)t]\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\left(p_1(x)\right)^{n-k}t^{n+k}$$

$$=[(a+b)(1-p_1(x)t)+(2a+1)t]\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(n+k)!}{k!(n)!}\left(p_1(x)\right)^{n}t^{n+2k}$$

On equating coefficient of t^n on both sides, we have

$$h_n(x)=(a+b)\left\{\sum_{k=0}^{[n/2]}\binom{n-k}{k}\left(p_1(x)\right)^{n-2k}-\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}\left(p_1(x)\right)^{n-2k-1}\right\}$$
$$+(2a+1)\sum_{k=0}^{[n/2]}\binom{n-k-1}{k}\left(p_1(x)\right)^{n-2k-1}.$$

The following theorems give the identity for generalized Fibonacci polynomial, known as Catalan's identity, Cassini's identity and, d'Ocagne's identity [6].

Theorem 2.6. Catalan's Identity:

Let $h_n\left(x\right)$ be the n^th^ term of generalized Fibonacci polynomial, then

$$h_n^2(x)-h_{n+r}(x)h_{n-r}(x)=\left(-1\right)^{n-r+1}\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}\left(p_2(x)\right)^{n-r}\left\{h_r(x)\right\}^2$$

Proof: Binet's formula for generalized Fibonacci polynomial is given by

$$h_n\left(x\right)=\left(A\alpha^n+B\beta^n\right)$$

Now,

$$h_n^2(x)-h_{n+r}(x)h_{n-r}(x)=\Big[\,(A\alpha^n+B\beta^n)\Big]^2-(A\alpha^{n+r}+B\beta^{n+r})(A\alpha^{n-r}+B\beta^{n-r})$$

$$=\Big\lfloor\,(A^2\alpha^{2n}+B^2\beta^{2n}+2AB\alpha^n\beta^n)\Big\rfloor-(A^2\alpha^{2n}+AB\alpha^{n+r}\beta^{n-r}+AB\alpha^{n-r}\beta^{n+r}+B^2\beta^{2n})$$

$$=AB\left(\alpha\beta\right)^n\left\{2-\alpha^r\beta^r-\alpha^{-r}\beta^{-r}\right\}$$

$$=-AB\left(\alpha\beta\right)^{n-r}\left(\alpha^r-\beta^r\right)^2$$

$$=-\left\{\frac{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)}{(\alpha-\beta)^2}\right\}\left(\alpha\beta\right)^{n-r}(\alpha^r-\beta^r)^2$$

$$=-\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}\left(-p_2(x)\right)^{n-r}\left\{\frac{(\alpha^r-\beta^r)}{(\alpha-\beta)}\right\}^2$$

$$=\left(-1\right)^{n-r+1}\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}\left(p_2(x)\right)^{n-r}\left\{h_r(x)\right\}^2.$$

Remark: For $p_1\left(x\right)=p_2\left(x\right)=1, a=0, b=0$, we obtain Catalan's identity for classical Fibonacci sequence which is stated in [6, Theorem 5.9].

Theorem 2.7. Cassini's Identity:

Let $h_n\left(x\right)$ be the n^th^ term of generalized Fibonacci polynomial, then

$$h_n^2(x)-h_{n+1}(x)h_{n-1}(x)=\left(-1\right)^n\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}\left(p_2(x)\right)^{n-1}\left\{2a+1\right\}^2$$

Proof: on substituting $r=1$ in Catalan's identity, we get the required result.

Theorem 2.8. d'Ocagne's Identity:

If the nth term of generalized Fibonacci polynomial is $h_n(x)$, then

$$h_m(x)h_{n+1}(x)-h_{m+1}(x)h_n(x)=\left(-1\right)^{n+1}\left(p_2(x)\right)^n\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}h_{m-n}(x)$$

Proof: Binet's formula for generalized Fibonacci polynomial is given by

$$h_n\left(x\right)=\left(A\alpha^n+B\beta^n\right)$$

Now,

$$h_m(x)h_{n+1}(x)-h_{m+1}(x)h_n(x)=(A\alpha^m+B\beta^m)(A\alpha^{n+1}+B\beta^{n+1})-(A\alpha^{m+1}+B\beta^{m+1})(A\alpha^n+B\beta^n)$$

$$=AB(\alpha^m\beta^{n+1}+\alpha^{n+1}\beta^m-\alpha^{m+1}\beta^n-\alpha^n\beta^{m+1})$$

$$=AB\left(\alpha\beta\right)^n\left\lfloor\beta\left(\alpha^{m-n}-\beta^{m-n}\right)-\alpha\left(\alpha^{m-n}-\beta^{m-n}\right)\right\rfloor$$

$$=-\left\{\frac{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)}{(\alpha-\beta)^2}\right\}\left(-p_2(x)\right)^n\left[\left(\alpha^{m-n}-\beta^{m-n}\right)\left(\alpha-\beta\right)\right]$$

$$=\left(-1\right)^{n+1}\left(p_2(x)\right)^n\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}\frac{\left(\alpha^{m-n}-\beta^{m-n}\right)}{\left(\alpha-\beta\right)}$$

$$=\left(-1\right)^{n+1}\left(p_2(x)\right)^n\left\{(2a+1)(a+b)p_1(x)-(2a+1)^2+(a+b)^2p_2(x)\right\}h_{m-n}(x).$$

Remark: We obtain d'Ocagne's identity for classical Fibonacci sequence by substituting $p_1\left(x\right)=p_2\left(x\right)=1,a=0,b=0$.

Similarly, Divisibility properties and many more identities, which were proved for classical Fibonacci Polynomial in [2] can also be derived for our generalized Fibonacci polynomial on same lines.

3. Some Identities of Generalized Lucas Polynomial

The results which we have proved in theorem 2.1 to theorem 2.5 can also be proved on similar lines for our generalized Lucas polynomial. Next, we explore the Lucas counterparts of Catalan's identity which have been stated for Fibonacci in [6].

Theorem 3.1

Let $t_n\left(x\right)$ be the nth term of generalized Lucas polynomial, then

$$t_n^2(x)-t_{n+r}(x)t_{n-r}(x)=\left(-q_2\left(x\right)\right)^{n-r}\left\{\frac{a^2\left(-q_2\left(x\right)\right)^r}{2}-\frac{a}{2}t_{2r}(x)\right\}$$

Proof: Binet's formula for Lucas polynomial is given by

$$t_n\left(x\right)=K\left(\alpha^n+\beta^n\right)$$

Now,

$$t_n^2(x)-t_{n+r}(x)t_{n-r}(x)=\Big[\,K(\alpha^n+\beta^n)\Big]^2-K(\alpha^{n+r}+\beta^{n+r})K(\alpha^{n-r}+\beta^{n-r})$$

$$=\Big\lfloor\,K^2(\alpha^{2n}+\beta^{2n}+2\alpha^n\beta^n)\Big\rfloor-K^2(\alpha^{2n}+\alpha^{n+r}\beta^{n-r}+\alpha^{n-r}\beta^{n+r}+\beta^{2n})$$

$$=2K^2\left(-q_2\left(x\right)\right)^n-K^2\left(-q_2\left(x\right)\right)^{n-r}(\alpha^{2r}+\beta^{2r})$$

$$=2K^2\left(-q_2\left(x\right)\right)^n-K\left(-q_2\left(x\right)\right)^{n-r}t_{2r}(x)$$

$$=\left(-q_2\left(x\right)\right)^n\left\{2K^2-K\left(-q_2\left(x\right)\right)^{-r}t_{2r}(x)\right\}$$

$$=\left(-q_2\left(x\right)\right)^{n-r}\left\{\frac{a^2\left(-q_2\left(x\right)\right)^r}{2}-\frac{a}{2}t_{2r}(x)\right\}.$$

The following theorem gives the identity for Lucas polynomial which is already derived for generalized Fibonacci polynomial, known as d'Ocagne's identity [6].

Theorem 3.2

If the n$^{\text{th}}$ term of generalized Lucas polynomial is $t_n(x)$, then

$$t_m(x)t_{n+1}(x)-t_{m+1}(x)t_n(x)=\frac{a}{2}\left\{\left(-q_2\left(x\right)\right)^{n+1}t_{m-n-1}(x)-\left(-q_2\left(x\right)\right)^{m+1}t_{n-m-1}(x)\right\}$$

Proof: Binet's formula for Lucas polynomial is given by

$$t_n\left(x\right)=K\left(\alpha^n+\beta^n\right)$$

Now,

$$t_m(x)t_{n+1}(x)-t_{m+1}(x)t_n(x)=K(\alpha^m+\beta^m)K(\alpha^{n+1}+\beta^{n+1})-K(\alpha^{m+1}+\beta^{m+1})K(\alpha^n+\beta^n)$$

$$=K^2(\alpha^m\beta^{n+1}+\alpha^{n+1}\beta^m-\alpha^{m+1}\beta^n-\alpha^n\beta^{m+1})$$

$$=K^2\left\{\left(\alpha\beta\right)^{n+1}(\alpha^{m-n-1}+\beta^{m-n-1})-\left(\alpha\beta\right)^{m+1}(\alpha^{n-m-1}+\beta^{n-m-1})\right\}$$

$$=\frac{a}{2}\left\{\left(-q_2\left(x\right)\right)^{n+1}t_{m-n-1}(x)-\left(-q_2\left(x\right)\right)^{m+1}t_{n-m-1}(x)\right\}.$$

The next theorem gives relevant result to Theorem 3.1 and 3.2 for our Lucas polynomial.

Theorem 3.3

Let $t_n\left(x\right)$ be the n^th^ term of generalized Lucas polynomial, then

$$t_n^2(x)+t_{n+r}(x)t_{n-r}(x)=a\,t_{2n}(x)+\frac{a^2}{2}\left(-q_2\left(x\right)\right)^n+\frac{a}{2}\left(-q_2\left(x\right)\right)^{n-r}t_{2r}(x) \tag{i}$$

$$t_m(x)t_{n+1}(x)+t_{m+1}(x)t_n(x)=\frac{a}{2}\left\{2t_{m+n+1}\left(x\right)+\left(-q_2\left(x\right)\right)^{n+1}t_{m-n-1}\left(x\right)+\left(-q_2\left(x\right)\right)^{m+1}t_{n-m-1}\left(x\right)\right\} \tag{ii}$$

Proof (i): With the help of Binet's formula we have,

$$t_n^2(x)+t_{n+r}(x)t_{n-r}(x)=\left[K(\alpha^n+\beta^n)\right]^2+K(\alpha^{n+r}+\beta^{n+r})K(\alpha^{n-r}+\beta^{n-r})$$

$$=\left[K(\alpha^{2n}+\beta^{2n}+2\alpha^n\beta^n)\right]^2+K^2(\alpha^{2n}+\alpha^{n+r}\beta^{n-r}+\alpha^{n-r}\beta^{n+r}+\beta^{2n})$$

$$=2K^2(\alpha^{2n}+\beta^{2n})+2K^2(-q_2\left(x\right))^n+K^2(-q_2\left(x\right))^{n-r}(\alpha^{2r}+\beta^{2r})$$

$$=2Kt_{2n}(x)+2K^2\left(-q_2\left(x\right)\right)^n+K\left(-q_2\left(x\right)\right)^{n-r}t_{2r}(x)$$

$$=at_{2n}(x)+\frac{a^2}{2}\left(-q_2\left(x\right)\right)^n+\frac{a}{2}\left(-q_2\left(x\right)\right)^{n-r}t_{2r}(x).$$

(ii): With the help of Binet's formula we have,

$$t_m(x)t_{n+1}(x)+t_{m+1}(x)t_n(x)=K(\alpha^m+\beta^m)K(\alpha^{n+1}+\beta^{n+1})+K(\alpha^{m+1}+\beta^{m+1})K(\alpha^n+\beta^n)$$

$$=K\left\{2K\left(\alpha^{m+n+1}+\beta^{m+n+1}\right)+\left(-q_2\left(x\right)\right)^{n+1}K\left(\alpha^{m-n-1}+\beta^{m-n-1}\right)+\left(-q_2\left(x\right)\right)^{m+1}K\left(\alpha^{n-m-1}+\beta^{n-m-1}\right)\right\}$$

$$=K\left\{2t_{m+n+1}(x)+\left(-q_2\left(x\right)\right)^{n+1}t_{m-n-1}(x)+\left(-q_2\left(x\right)\right)^{m+1}t_{n-m-1}(x)\right\}$$

$$=\frac{a}{2}\left\{2t_{m+n+1}\left(x\right)+\left(-q_2\left(x\right)\right)^{n+1}t_{m-n-1}\left(x\right)+\left(-q_2\left(x\right)\right)^{m+1}t_{n-m-1}\left(x\right)\right\}.$$

4. Conclusion

In this paper, we have stated and derived many properties of generalized Fibonacci polynomial through the generating function and Binet's formula and generalized Lucas polynomial through Binet's formula. Many other identities can be derived using our generalized Fibonacci and Lucas polynomial.

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