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STUDY OF η-EINSTEIN SOLITON ON $(LCS)_n$-MANIFOLD

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Abstract: The object of the present paper is to study η-Einstein soliton on $(LCS)_n$-manifold. We have also studied $(LCS)_n$-manifold admitting η-Einstein soliton where the Ricci tensors are cyclic parallel.

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1. Introduction

The notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) was first introduced in 2003 by Shaikh [18] with an example that generalizes the notion of LP-Sasakian manifolds which was introduced by Matsumoto [11] and also by Mihai and Rosca [12]. In 2005 and 2006, the application of $(LCS)_n$-manifolds to the general theory of relativity and cosmology was investigated by Shaikh and Baishya [21], [22]. Later many other authors like Atceken ([1], [2]), Hui ([8], [9], [10]), Narain [13], Yadav ([23], [24]), Roy, Dey, Bhattacharyya [17], Shaikh ([19], [20]) also studied the $(LCS)_n$-manifold.

In 2016, Catino and Mazzieri [4] introduced the notion of Einstein soliton which can be viewed as a self-similar solution to the Einstein flow

$$\frac{\partial g}{\partial t} = -2(S - \frac{r}{2} g)$$

where g is the Riemannian metric, S is the Ricci tensor and r is the scalar curvature. The Einstein soliton plays an important role in solving many physical and geometrical problems. The Einstein soliton is analogue to the Ricci soliton which is also generated by a self-similar solution to the very famous geometric revolution equation Ricci flow which was used by Perelman([15], [16]) to prove Poincare conjecture. A slight deviation of the Einstein soliton, called the η-Einstein soliton is defined by the following mathematical expression,

$$\pounds_\xi g + 2S + (2\lambda - r)g + 2\mu\eta \otimes \eta = 0 \tag{1}$$

where $\pounds_\xi$ denotes the Lie derivative along the direction of the vector field ξ, S is the Ricci tensor, r is the scalar curvature and λ, μ are real constants. The η-Einstein soliton is called shrinking, steady and expanding according as $\lambda < 0$, $= 0$ and > 0. In particular if $\mu = 0$, the η-Einstein soliton reduces to the Einstein soliton (g, ξ, λ).

A more general notion of a Ricci soliton is that of a η-Ricci soliton introduced by Cho and Kimura [7], which was treated by C. Calin and M. Crasmareanu on Hopf surfaces in complex space-forms [3]. The η-Einstein soliton reduces to the η-Ricci soliton if we consider the manifold to have constant scalar curvature.

This paper is organized as follows: Section 2 is concerned with preliminaries. Section 3 deals with $(LCS)_n$-manifold admitting η-Einstein soliton. In section 4, we have investigated the nature of the $(LCS)_n$-manifold admitting η-Einstein soliton where the Ricci tensors are cyclic parallel, η-recurrent cyclic parallel and for the Ricci symmetric manifold. In section 5, we have established an example to verify the results we obtained.

2. Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \epsilon M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature (-,+,...,+), where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector field $v \epsilon T_pM$ is said to be timelike if it satisfies $g_p(v, v) < 0$ [14].

Definition 2.1 ([23]) In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X) \tag{2}$$

for any $X \epsilon \Gamma(TM)$, the section of all smooth tangent vector fields on M , is said to be a concircular vector field if

$$(\nabla_X A)Y = \alpha \{g(X, Y) + \omega(X)A(Y)\},$$

where A is defined in the earlier equation, α is a non-zero scalar and ω is a closed 1-form and ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

Definition 2.2 A vector field ξ is called torse-forming if it satisfies (2) $\nabla_X \xi = f X + \gamma(X)\xi$

for a smooth function $f \epsilon C^\infty(M)$ and γ is a 1-form, for all vector fields X on

M . A torse-forming vector field ξ is called recurrent if $f = 0$.

Definition 2.3 A vector field v is called concurrent vector field if it satisfies

$$\nabla_X v = 0 \tag{3}$$

for any vector field X on M .

Definition 2.4 A tensor h of second order is said to be a parallel tensor if $\nabla h = 0$.

If the Lorentzian manifold M admits a unit timelike concircular vector field

ξ, called the generator of the manifold, then we have,

$$g(\xi, \xi) = -1,\ g(X, \xi) = \eta(X),\ (\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \tag{4}$$

where $\alpha \neq 0$ and η is a non-zero 1-form. From (4), (5)

$$\nabla_X \xi = \alpha\{X + \eta(X)\xi\} \tag{5}$$

for all vector fields X on M and α satisfies

$$(\nabla_X \alpha) = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{6}$$

ρ being a certain scalar function given by

$$\rho = -(\xi\alpha).$$

If we put

$$\alpha\phi X = \nabla_X \xi, \tag{7}$$

then (5) and (7) give,

$$\phi X = X + \eta(X)\xi, \tag{8}$$

where ϕ is a (1,1)-tensor, called the structure tensor of M . Thus the Lorentzian manifold M together with a unit timelike concircular vector field ξ, its associated 1-form η and (1,1)-tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold) [18]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [11].

In a $(LCS)_n$ manifold ($n > 2$), the following relations hold([24], [19], [21], [22]):

$$\eta(\xi) = -1,\ \phi\xi = 0,\ \eta(\phi X) = 0,\ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \tag{9}$$

$$\phi^2 X = X + \eta(X)\xi \tag{10}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X) \tag{11}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y] \tag{12}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y] \tag{13}$$

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X] \tag{14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{15}$$

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\}\xi \tag{16}$$

for any vector fields X, Y, Z on M and

$$\beta = -(\xi\rho)$$

is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

Definition 2.5 A $(LCS)_n$-manifold (M^n, g) is said to be η-Einstein if its Ricci tensor S of type (0, 2) is of the form

$$S = ag + b\eta \otimes \eta, \tag{17}$$

where a and b are smooth functions on M .

In an η-Einstein $(LCS)_n$-manifold, the following relations hold [10]:

$$S(\phi X, Y) = S(X, \phi Y) = ag(\phi X, Y), \tag{18}$$

$$S(X, \xi) = (a - b)\eta(X), S(\xi, \xi) = -(a - b), \tag{19}$$

$$S(\phi X, \phi Y) = S(X, \phi^2 Y) = S(X, Y) + (a - b)\eta(X)\eta(Y). \tag{20}$$

All these definitions and results will be required in next sections.

3. η-Einstein Soliton on $(LCS)_n$-manifolds

Let (g, ξ, λ, μ) be an η-Einstein soliton on a $(LCS)_n$-manifold. Then in view of (4) and (5), we have from (1.1) that

$$S(X,Y) = \left(\frac{r}{2} - \alpha - \lambda\right) g(X,Y) - - (\alpha + \mu)\eta(X)\eta(Y), \tag{21}$$

which shows that the manifold (M, g) is an η-Einstein manifold. Hence we have,

Theorem 3.1. *If a $(LCS)_n$-manifold (M, g) admits an η-Einstein soli- ton (g, ξ, λ, μ), then the manifold becomes an η-Einstein manifold.*

Next we consider an η-Einstein soliton on an η-Einstein $(LCS)_n$-manifold.

Using (17) in (1) we have,

$$g(\nabla_X\xi, Y)+g(X, \nabla_Y \xi)+(2a+2\lambda-r)g(X, Y)+2(b+\mu)\eta(X)\eta(Y) = 0. \tag{22}$$

Putting $X = Y = \xi$ in (22) and using (9) we have

$$2g(\nabla_\xi\xi, \xi) - 2a + 2b + (r - 2\lambda) + 2\mu = 0$$

which yields, $g\left(\nabla_\xi \xi, \xi\right) = (a - b) - (\frac{r}{2} - \lambda + \mu)$

But we know, $g(\nabla_X\xi, \xi) =0$ for any vector field X on M , since ξ has a constant norm. So, we get,

$$(a - b) = (\frac{r}{2} - \lambda + \mu)$$

Consequently, (22) becomes,

$$g(\nabla_X\xi, Y) + g(X, \nabla_Y \xi) + 2(b + \mu)g(\phi X, \phi Y) = 0. \tag{23}$$

Replacing Y by ξ in (23), we have,

$$g(\nabla_\xi\xi, X) = 0$$

for any vector field X on M .

So, $\nabla_\xi\xi = 0$, i.e., ξ is a geodesic vector field. Setting $X = Y = \xi$ in (14) we have,

$(\nabla_\xi \phi)\xi = 0.$

Setting $X = \xi$ in (4), we have

$\nabla_\xi \eta = 0.$

Thus the above discussion leads to the following:

Theorem 3.2. *If* (g, ξ, λ, μ) *be an* η*-Einstein soliton on a* $(LCS)_n$*-manifold, then*

i) $(a - b) = (\frac{r}{2} - \lambda + \mu)$

ii) ξ *is a geodesic vector field*

iii) $(\nabla_\xi \phi)\xi = 0$ *and* $\nabla_\xi \eta = 0$.

Now, let (g, ξ, λ, μ) be an η-Einstein soliton on a $(LCS)_n$-manifold M where ξ is a torse-forming vector field. Then we have from (2) that

$g(\nabla_X \xi, \xi) = f\eta(X) - \gamma(X)$

which gives $\gamma = f\eta$

and consequently (2) becomes.

$$\nabla_X \xi = f\,[X + \eta(X)\xi] \tag{24}$$

Using (24) in (23), we have,

$$(f + b + \mu)g(\phi X, \phi Y) = 0 \tag{25}$$

for all vector fields X and Y. This gives, $f = -(b + \mu)$. It follows that,

$$\nabla_X \xi = -(b + \mu)[X + \eta(X)\xi]. \tag{26}$$

From (10) it implies that $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X and hence we get $d\eta = 0$, i.e. η is closed.

It is known that

$$R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{27}$$

With the help of (26), (27) yields

$$R(X, Y)\xi = (b + \mu)^2[\eta(Y)X - \eta(X)Y] \tag{28}$$

and hence in view of theorem 3.2(i) we get,

$$S(X, \xi) = (n - 1)\left(a - \frac{r}{2} + \lambda\right)^2 \eta(X) \tag{29}$$

Comparing with (19) we get,

$b = a - (n - 1)(a - \frac{r}{2} + \lambda)^2$

and $\mu = \left(\lambda - \frac{r}{2}\right) + (n-1)(\, a - \frac{r}{2} + \lambda)^2$

This leads to the following theorem:

Theorem 3.3. *Let* (g, ξ, λ, μ) *be an* η*-Einstein soliton on a* $(LCS)_n$*-manifold M where* ξ *is a torse-forming vector field. Then*

i) $f = -(b + \mu)$,

ii) η *is closed,*

iii) $b = a - (n-1)(\, a - \frac{r}{2} + \lambda)^2$

and $\mu = \left(\lambda - \frac{r}{2}\right) + (n-1)(\, a - \frac{r}{2} + \lambda)^2$

If we consider ξ to be recurrent, then $f = 0$ and therefore, $b + \mu = 0$, which gives $\nabla_X \xi = 0$ for all vector fields X on M and hence ξ is a concurrent vector field.

Also in that case,

$(\pounds_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$

which means that ξ is also a Killing vector field. Hence we have,

Corollary 3.1. *Let* (g, ξ, λ, μ) *be an* η*-Einstein soliton on a* $(LCS)_n$*-manifold M where* ξ *is a torse-forming vector field. Then* ξ *is concurrent and Killing vector field.*

Next we consider a $(LCS)_n$-manifold (M, g) which admits an η-Einstein soliton (g, V, λ, μ) such that V is pointwise collinear with ξ i.e., $V = b\xi$, where b is a function on the manifold. Then from the equation (1) it follows that

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(\nabla_Y \xi, X) + (Yb)\eta(X) \quad (30)$$
$$2S(X, Y) + (2\lambda - r)g(X, Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (5) in above we have,

$$(2b\alpha + 2\lambda - r)g(X, Y) + 2(b\alpha + \mu)\eta(X)\eta(Y) + (Xb)\eta(Y) \quad (31)$$
$$+(Yb)\eta(X) + 2S(X, Y) = 0.$$

Substituting Y by ξ, we have,

$$-(Xb) + (\xi b)\eta(X) = 0 \quad (32)$$

Again replacing X by ξ, we have,

$$\xi b = 0, \quad (33)$$

which yields,

$$Xb = 0, \quad (34)$$

which implies that b is constant. So, we can state the following theorem:

Theorem 3.4. *Let M be a $(LCS)_n$-manifold admitting an η-Einstein soliton $(g, V, \xi, \lambda, \mu)$, V being a vector field on M. If V is pointwise collinear with ξ, then V is a constant multiple of ξ, ξ being the Reeb vector field of M.*

Next we consider that the symmetric tensor field $h = \pounds_\xi g + 2S - rg + 2\mu\eta \otimes \eta$ of type (0,2) is parallel with respect to the Levi-Civita connection ∇. Then, using (5) we have,

$$h(X, Y) = 2\alpha g(X, Y) + 2\alpha\eta(X)\eta(Y) + 2S(X, Y) \tag{35}$$

$$-rg(X, Y) + 2\mu\eta(X)\eta(Y).$$

Replacing X and Y by ξ in the above equation, we get,

$$h(\xi, \xi) = 2S(\xi, \xi) + r + 2\mu$$

Using (21), we have,

$$h(\xi, \xi) = 2\lambda$$

and, therefore, $\lambda = {}^{1}\, h(\xi, \xi).$

From [5] and [6], we have

$$h(X, Y) = -h(\xi, \xi)g(X, Y), \qquad \forall X, Y \epsilon \chi(M)$$

Thus, $\pounds_\xi g + 2S - rg + 2\mu\eta \otimes \eta = -2\lambda g$

which gives an η-Einstein soliton. This leads to the following theorem:

Theorem 3.5. *Let $(M, g, \xi, \lambda, \mu)$ be a $(LCS)_n$-manifold. If the symmetric tensor field $h = \pounds_\xi g + 2S - rg + 2\mu\eta\otimes\eta$ of type (0,2) is parallel with respect to the Levi-Civita connection∇ , then (g, ξ, λ) admits an η-Einstein soliton.*

Corollary 3.2. *On a $(LCS)_n$-manifold $(M, g, \xi, \lambda, \mu)$ with the property that a symmetric tensor field $h = \pounds_\xi g + 2S$ of type (0,2) is parallel with respect to the Levi-Civita connection associated to g, then the equation with $r = 2\lambda = 2(n - 1)(\alpha^2 - \rho)$ and $\mu = 0$ define a Ricci soliton.*

From the equation (21), we have,

$$(\nabla_X S)(Y, Z) = -(\alpha + \mu)[\eta(Y)g(Z, \nabla_X\xi) + \eta(Z)g(Y, \nabla_X\xi)], \tag{36}$$

$$(\nabla_X Q)Y = -(\alpha + \mu)[\eta(Y)\nabla_X\xi + g(Y, \nabla_X\xi)\xi] \tag{37}$$

Putting $X = \xi$ in (36) and using (5) we have,

$$(\nabla_\xi S)(X, Y) = 0$$

for any vector fields X, Y on M . Hence S is parallel along ξ. Again replacing X by ξ in (37) and using (5) we have,

$$(\nabla_\xi Q)X = 0$$

for any vector field X on M . Hence Q is parallel along ξ. Thus we can state the following theorem:

Theorem 3.6. *Let M be a $(LCS)_n$-manifold admitting an η-Einstein soliton (g, ξ, λ, μ), ξ being the Reeb vector field on M. Then Q and S are parallel along ξ, where Q is the Ricci operator, defined by $S(X, Y) = g(QX, Y)$ and S is the Ricci tensor of M.*

4. η-Einstein Soliton on $(LCS)_n$-manifold with Cyclic Parallel Ricci Tensor

Let M be a $(LCS)_n$-manifold admitting an η-Einstein soliton (g, ξ, λ, μ). If possible, we suppose that the Ricci tensor S of M is cyclic parallel, that is,

$$(\nabla_X S)(Y, Z)+(\nabla_Y S)(Z, X)+(\nabla_Z S)(X, Y) = 0 \qquad \forall X, Y, Z \epsilon \chi(M). \tag{38}$$

Using (36) we have,

$$(\alpha + \mu)[\eta(Y)g(Z, \nabla_X\xi) + \eta(Z)g(Y, \nabla_X\xi) + \eta(Z)g(X, \nabla_Y\xi) \tag{39}$$
$$+\eta(X)g(Z, \nabla_Y\xi) + \eta(X)g(Y, \nabla_Z\xi) + \eta(Y)g(X, \nabla_Z\xi)] = 0$$

Using (5) we have,

$$\alpha(\alpha + \mu)[g(X, Z)\eta(Y) + g(X, Y)\eta(Z) + g(Y, Z)\eta(X) + 3\eta(X)\eta(Y)\eta(Z)] = 0 \tag{40}$$

which arises three cases.

If

$$\alpha + \mu = 0, \tag{41}$$

then using (41) in (21) we have,

$$S(X,Y) = \left(\frac{r}{2} - \alpha - \lambda\right) g(X,Y) \tag{42}$$

which shows that the manifold is Einstein.

If $\alpha = 0$, then $\nabla_X\xi = 0$, which gives $(\pounds_\xi g)(X, Y) = 0$ for any vector fields X, Y on M and hence ξ is a Killing vector field.

If we consider, $g(X, Z)\eta(Y)+g(X, Y)\eta(Z)+g(Y, Z)\eta(X)+3\eta(X)\eta(Y)\eta(Z) = 0$, then replacing $Z = \xi$, we have,

$$g(X, Y) + \eta(X)\eta(Y) = 0, \tag{43}$$

which means

$$g(\phi X, \phi Y) = 0$$

which is impossible. Thus we conclude the following theorem:

Theorem 4.1. *Let (g, ξ, λ, μ) be an η-Einstein soliton on a $(LCS)_n$-manifold (M, g). If the Ricci tensor S of M is cyclic parallel, then either the manifold is Einstein or ξ is a Killing vector field.*

Next we suppose that M is η-recurrent, that is,

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z) \qquad \forall X, Y, Z \epsilon \chi(M).$$

Moreover, if M is cyclic parallel, then using (38) we have,

$$\eta(X)S(Y, Z) + \eta(Y)S(Z, X) + \eta(Z)S(X, Y) = 0. \tag{44}$$

With the help of (21), (44) yields,

$$(\frac{r}{2} - \alpha - \lambda)[g(Y, Z)\eta(X)+g(X, Z)\eta(Y)+g(X, Y)\eta(Z)]-3(\alpha+\mu)\eta(X)\eta(Y)\eta(Z) = 0. \tag{45}$$

Putting $Y = Z = \xi$ in (4.8), we have,

$$(\frac{r}{2} - \lambda + \mu)\eta(X) = 0 \tag{46}$$

which in turn gives, $a - b = 0$, which is a contradiction. This leads to the following theorem:

Theorem 4.2. *Let (g, ξ, λ, μ) be an η-Einstein soliton on a $(LCS)_n$-manifold (M, g). If the Ricci tensor S of M is cyclic parallel η-recurrent, then there does not exist an η-Einstein soliton with the potential vector field ξ on M.*

If the $(LCS)_n$-manifold M is Ricci symmetric, then

$$(\nabla_X S)(Y, Z) = 0. \tag{47}$$

Using (36) we have,

$$(\alpha + \mu)[\eta(Y)g(Z, \nabla_X\xi) + \eta(Z)g(Y, \nabla_X\xi)] = 0. \tag{48}$$

Replacing Z by ξ, we have

$$(\alpha + \mu)g(\nabla_X\xi, Y) = 0 \tag{49}$$

which gives, $\alpha + \mu = 0$ and hence the manifold becomes Einstein. Thus we conclude the following theorem:

Theorem 4.3. *Let (g, ξ, λ, μ) be an η-Einstein soliton on a $(LCS)_n$-manifold (M, g). If we consider the manifold to be Ricci symmetric, then the manifold becomes Einstein.*

5. Example

We refer the example of $(LCS)_n$-manifold constructed by Roy et al. [17].

Consider the 3-dimensional manifold $M = (x, y, z)\epsilon\mathbb{R}3$, $z \neq 0$ where (x, y, z) are standard co-ordinates in $\mathbb{R}3$ and let us take the vector fields

$$e_1 = z\frac{\partial}{\partial z}, e_2 = z\frac{\partial}{\partial y}, e_3 = z\frac{\partial}{\partial z}$$

which are linearly independent at each point of M . Let g be

the Riemannian metric defined by,

$g(e_1, e_1) = g(e_2, e_2) = 1$, $g(e_3, e_3) = -1$

$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0$

and the 1-form η is defined by $\eta(z) = g(z, e_3)$ for any vector fields Z on M

and the (1,1) tensor field ϕ is defined by

$\phi e_1 = e_2$, $\phi e_2 = e_1$, $\phi e_3 = 0$.

For $e_3 = \xi$, using Koszul's formula, we have,

$$\nabla_{e_1} e_3 = -e_1, \nabla_{e_2} e_3 = -e_2, \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_2} e_1 = 0, \nabla_{e_3} e_1 = 0,$$

$$\nabla_{e_1} e_2 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_3} e_2 = 0.$$

Consequently for $e_3 = \xi$, $(M, g, \xi, \eta, \phi, \alpha)$ is a $(LCS)_n$-manifold of dimension 3, where $\alpha = 1$.

Using the above relations, we can easily calculate the non-vanishing components of the curvature tensor R and the Ricci tensor S as follows:

$R(e_2, e_3)e_3 = -e_2$, $R(e_1, e_3)e_3 = -e_1$, $R(e_1, e_2)e_2 = e_1$, $R(e_2, e_3)e_2 = -e_3$, $R(e_1, e_3)e_1 = -e_3$, $R(e_1, e_2)e_1 = -e_2$, $S(e_1, e_1) = -2$, $S(e_2, e_2) = -2$, $S(e_3, e_3) = -2$

Also from (21), we have

$$S(e_{1,}e_1) = \left(\frac{r}{2} - \alpha - \lambda\right), S(e_{2,}e_2) = \left(\frac{r}{2} - \alpha - \lambda\right), S(e_{3,}e_3) = \left(\frac{r}{2} - \alpha - \lambda\right) - (\alpha + \mu)$$

Comparing we have,

$\left(\frac{r}{2} - \alpha - \lambda\right) = -2$ *and* $2 - (\alpha + \mu) = -2$,

which gives, $\left(\frac{r}{2} - \lambda + \mu\right) = 2$.

Now from (17), we calculate that,

$S(e_1, e_1) = a$, $S(e_2, e_2) = a$, $S(e_3, e_3) = -a + b$

Comparing again we have, $a = -2$, $a - b = 2$, which results, $b = -4$ and this verifies theorem 3.2.

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