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CERTAIN CURVATURE PROPERTIES ON GENERALIZED SASAKIAN-SPACE-FORMS

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Abstract: In the present paper we study ξ-quasi-conformally flat, ξ-pseudo-projectively flat, ϕ-quasi-conformally semi-symmetric and ϕ-pseudo-projective semi-symmetric generalized Sasakian-space-forms. Condition for $(2n+1)$-dimensional generalized Sasakian-space-forms to be $\mathbb{Q}$-curvature pseudo-symmetric is also the part of paper.

Keywords: Generalized Sasakian-space-forms, ξ-quasi-conformally flat, ϕ-quasi-conformally semi-symmetric, ξ-pseudo-projective flat, ϕ-pseudo-projective semi-symmetric, $\mathbb{Q}$-curvature pseudo-symmetric.

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1. Introduction

Notation of generalized Sasakian-space-forms were introduced by Alegre et al. [1]. A Sasakian manifold with constant ϕ-sectional curvature is a Sasakian-space-forms and it has a specific form of curvature tensor. An almost contact metric manifold $(M^{2n+1},\phi,\xi,\eta,g)$ is said to be generalized Sasakian-space-forms if there exists three differentiable functions f_1, f_2, f_3 on M such that the curvature tensor R is given by

$$\begin{aligned}R(X,Y)Z = \ & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z\} \\ & + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi\}, \qquad (1)\end{aligned}$$

for any vector fields X, Y, Z on M^{2n+1}. In such case we denote the manifold as $M^{2n+1}(f_1, f_2, f_3)$. In [1] the authors gives several examples of generalized Sasakian-space-forms. If $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes ϕ-sectional curvature, then a generalized Sasakian-space-forms with Sasakian structure becomes a Sasakian-space-forms.

A Riemannian manifold (M^{2n+1}, g) is called locally symmetric if it's curvature tensor R is parallel, i.e. $\nabla R = 0$, where ∇ is the Levi-Civita connection. The notation of semi symmetric, a proper generalization of locally symmetric manifold, is defined by $R(X,Y).R = 0$, where $R(X,Y)$ acts on R as derivation. A complete intrinsic classification of these manifolds was given by Szabo[11]. In [5] Kushwaha and Narain studied some curvature properties on Sasakian manifold. Also projective and conformal curvature tensor in K- contact η-Einstein manifolds have been studied by Kushwaha and Narain [6].

In a Riemannian manifold $M^{2n+1}(n > 1)$, the quasi – conformal curvature tensor W of type (1, 3) is defined by [12]

$$W(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$
$$- \frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}[g(Y,Z)X - g(X,Z)Y], \tag{2}$$

where a, b are constant such that $a, b \neq 0$ and r is the scalar curvature. In particular if $a = 1$ and $b = -\frac{1}{2n-1}$ then the quasi-conformal curvature tensor reduces to conformal curvature tensor. If $a = 1$ and $b = 0$ then the quasi-conformal curvature tensor reduces to concircular curvature tensor. Hence the name 'quasi-conformal' is justified.

Mantica and Suh [7] introduced a new curvature tensor of type (1,3) in a $(2n + 1)$-dimensional Riemannian manifold $(M^{2n+1}, g), n > 1$, denoted by $\mathbb{Q}$ and defined by

$$\mathbb{Q}(X,Y)Z = R(X,Y)Z - \frac{\Psi}{2n}[g(Y,Z)X - g(X,Z)Y], \tag{3}$$

where Ψ is an arbitrary scalar function. Such a tensor $\mathbb{Q}$ is known as $\mathbb{Q}$-curvature tensor. The notation of $\mathbb{Q}$ tensor is also suitable to interpret again some differential structure on a Riemannian manifold. If $\Psi = \frac{r}{2n+1}$, then $\mathbb{Q}$-curvature tensor reduces to concircular curvature tensor.

Further in a $(2n + 1) -$dimensional almost contact metric manifold, the pseudo-projective curvature tensor $\widetilde{P}$ [8] is defined by

$$\tilde{P}(X,Y)Z = aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y]$$
$$- \frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}[g(Y,Z)X - g(X,Z)Y], \tag{4}$$

where a and b are constants and R, S and r are the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of manifold respectively. If $a = 1, b = -\frac{1}{2n}$ then (4) takes the form

$$\tilde{P}(X,Y)Z = P(X,Y)Z, \tag{5}$$

where P is the projective curvature tensor.

For a tensor field T of type $(0,k)$ on M, $k \geq 1$, and a symmetric tensor field A of type $(0,2)$ on M we define the $(0,k+2)$ tensor field $R.T$ and $Q(A,T)$ by

$$\begin{aligned}(R(X,Y).T)(X_1,X_2,\ldots\ldots\ldots\ldots X_k) = &-T(R(X,Y)X_1,X_2,\ldots\ldots\ldots..,X_k)\\ &- T(X_1,R(X,Y)X_2,\ldots\ldots\ldots..,X_K)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &-T(X_1,X_2,\ldots\ldots\ldots\ldots,R(X,Y)X_k)\end{aligned} \tag{6}$$

and $$\begin{aligned}Q(A,T)(X_1,X_2,\ldots\ldots\ldots\ldots X_k) = &-T\big((X\wedge_A Y)X_1,X_2,\ldots\ldots\ldots..,X_k\big)\\ &- T(X_1,(X\wedge_A Y)X_2,\ldots\ldots\ldots\ldots,X_K)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &-T(X_1,X_2,\ldots\ldots\ldots\ldots,(X\wedge_A Y)X_k),\end{aligned} \tag{7}$$

where $(X\wedge_A Y)$ is endomorphism given by

$$(X\wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{8}$$

A Riemannian manifold M is said to be pseudo-symmetric [3], [4] if

$$R.R = L_R Q(g,R) \tag{9}$$

holds on the set $U_R = \left\{x \in M{:}\, R - \frac{r}{n(n-1)}G \neq 0 \; at \; x\right\}$, where G is $(0,4)-$ tensor defined by $G(X_1,X_2,X_3,X_4) = g\big((X_1\wedge X_2)X_3,X_4\big)$ and L_R is some function on M. A Riemannian manifold M is said to be $\mathbb{Q}$-curvature pseudo-symmetric if

$$R.\mathbb{Q} = L_{\mathbb{Q}} Q(g,\mathbb{Q}) \tag{10}$$

holds on the set $U_{\mathbb{Q}}$= $\{x \in M{:}\, \mathbb{Q} \neq 0 \; at \; x\}$, where $L_{\mathbb{Q}}$ is some function on $U_{\mathbb{Q}}$ and $\mathbb{Q}$ is the $\mathbb{Q}$-curvature tensor.

An almost contact manifold M^{2n+1} is said to be η-Einstein if it's Ricci tensor S is of the form

$$S = Ag + B\eta \otimes \eta, \tag{11}$$

where A and B are constants, η is called the associated 1-form and vector field ξ defined by

$$g(X,\xi) = \eta(X), \tag{12}$$

is called the generator. If $B = 0$ then the manifold is Einstein and if $A = 0$ then the manifold is special type of η-Einstein.

The paper is organized as follows. Section 2 of present paper contains some preliminary results on generalized Sasakian-space-forms. In section 3 and 5, we study ξ-quasi-conformally flat and ξ-pseudo-projective flat generalized Sasakian-space-forms respectively. In section 4 and 6, we study ϕ-quasi conformally semi-symmetric and ξ-pseudo-projective semi-symmetric generalized Sasakian-space-forms respectively. The last section is devoted to study of $\mathbb{Q}$-curvature pseudo-symmetric generalized Sasakian-space-forms and proved that if a $(2n+1)$-dimensional generalized Sasakian-space-forms is $\mathbb{Q}$-curvature pseudo-symmetric then either $L_{\mathbb{Q}} = f_1 - f_3$ or the manifold is η-Einstein.

2. Preliminaries

A $(2n+1)$ dimensional Riemannian manifold $M(\phi, \xi, \eta, g)$, where ϕ is a tensor field, ξ is a contravariant vector field, η is a 1-form and g is a Riemannian metric, is called an almost contact metric manifold if the following results holds [2][9][10]:

$$\phi^2 X = -X + \eta(X)\xi, \phi\xi = 0, \quad (13)$$

$$\eta(\xi) = 1, g(X,\xi) = \eta(X), \ \eta(\phi X) = 0, \quad (14)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad (15)$$

$$g(\phi X, Y) = -g(X, \phi Y), g(\phi X, X) = 0, \quad (16)$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y). \quad (17)$$

For a $(2n+1)$-dimensional generalized Sasakian-space-forms [1], the curvature tensor R, the Ricci tensor S and Ricci operator Q satisfy

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}, \quad (18)$$

$$R(\xi,X)Y = (f_1 - f_3)\{g(X,Y)\xi - \eta(Y)X\}, \quad (19)$$

$$R(\xi,X)\xi = (f_1 - f_3)\{\eta(X)\xi - X\}, \quad (20)$$

$$S(X,Y) = (2nf_1 + 3f_2 - f_3)g(X,Y) - (3f_2 + (2n-1)f_3)\eta(X)\eta(Y), \quad (21)$$

$$S(X,\xi) = 2n(f_1 - f_3)\eta(X), \quad (22)$$

$$QX = (2nf_1 + 3f_2 - f_3)X - (3f_2 + (2n-1)f_3)\eta(X)\xi, \quad (23)$$

$$r = 2n(2n+1)f_1 + 6nf_2 - 4nf_3, \quad (24)$$

$$\eta(R(X,Y)Z) = (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \quad (25)$$

$$\eta(R(X,Y)\xi) = 0, \quad (26)$$

$$\eta(R(\xi,X)Y) = (f_1 - f_3)\{g(X,Y) - \eta(X)\eta(Y)\}, \quad (27)$$

$$S(\phi X, \phi Y) = S(X,Y) - 2n(f_1 - f_3)\eta(X)\eta(Y). \quad (28)$$

Further, for a $(2n+1)$-dimensional generalized Sasakian-space-forms, from (3) we have

$$\mathbb{Q}(X,Y)\xi = \left(f_1 - f_3 - \frac{\Psi}{2n}\right)\{\eta(Y)X - \eta(X)Y\}, \quad (29)$$

$$\mathbb{Q}(\xi,X)Y = \left(f_1 - f_3 - \frac{\Psi}{2n}\right)\{g(X,Y)\xi - \eta(Y)X\}, \quad (30)$$

$$\mathbb{Q}(\xi,X)\xi = \left(f_1 - f_3 - \frac{\Psi}{2n}\right)\{\eta(X)\xi - X\}, \quad (31)$$

$$\eta(\mathbb{Q}(X,Y)\xi) = \eta(\mathbb{Q}(X,\xi)\xi) = \eta(\mathbb{Q}(\xi,X)\xi) = 0. \quad (32)$$

3. ξ-quasi-conformally flat generalized Sasakian-space-forms

Definition 3.1: An almost contact metric manifold $(M^{2n+1}, g), n > 1$, is said to be ξ-quasi-conformally flat if it satisfies the condition

$$W(X,Y)\xi = 0$$

on M, for all $X, Y \in \chi(M)$.

Let M be a $(2n+1)$-dimensional ξ-quasi-conformally flat generalized Sasakian-space-forms then from (2), we have

$$aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY]$$

$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}[g(Y,\xi)X - g(X,\xi)Y] = 0. \quad (33)$$

Using (18), (22) and (23) in (33), we get

$$\left[a(f_1 - f_3) + b\big(2n(f_1 - f_3) + (2nf_1 + 3f_2 - f_3)\big) - \frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}\right]$$

$$[\eta(Y)X - \eta(X)Y] = 0. \quad (34)$$

Since $[\eta(Y)X - \eta(X)Y] \neq 0$, then from (34), we have

$$r = 2n(2n+1)\left[\frac{a(f_1 - f_3) + b(4nf_1 + 3f_2 - (2n+1)f_3)}{a + 2nb}\right]. \quad (35)$$

This leads to the following result:

Theorem 3.1: A $(2n+1)$-dimensional generalized Sasakian-space-forms is ξ-quasi-conformally flat if $r = 2n(2n+1)\left[\frac{a(f_1 - f_3) + b(4nf_1 + 3f_2 - (2n+1)f_3)}{a + 2nb}\right]$.

Taking $a = 1$ and $b = 0$ in (35), we get

$$r = 2n(2n+1)(f_1 - f_3). \quad (36)$$

This leads to the following result:

Corollary 3.1: A $(2n+1)$-dimensional generalized Sasakian-space-forms is ξ-concircularly flat if $r = 2n(2n+1)(f_1 - f_3)$.

Using (24) in (36), we get

$$f_3 = \frac{3f_2}{(1-2n)}. \quad (37)$$

This leads to the following result:

Corollary 3.2: A $(2n+1)$-dimensional generalized Sasakian-space-forms is ξ-concircularly flat if $f_3 = \frac{3f_2}{(1-2n)}$.

From (21) and (37), we have

$$S(X,Y) = 2n\left(f_1 - \frac{3f_2}{(1-2n)}\right)g(X,Y). \tag{38}$$

This leads to the following result:

Corollary 3.3: A ξ-concircularly flat generalized Sasakian-space-forms is an Einstein manifold.

Again taking $a = 1$ and $b = -\frac{1}{(2n-1)}$ in (35), we get $r = 0$.

This leads to the following result:

Corollary 3.4: A $(2n+1)$-dimensional generalized Sasakian-space-forms is ξ-conformally flat if $r = 0$.

5. ϕ- $q$uasi-conformally semi-symmetric generalized Sasakian space form

Definition 4.1: An almost contact metric manifold $(M^{2n+1}, g), n > 1$, is said to be ϕ- $q$uasi- conformally semi-symmetric if it satisfies the condition

$$W(X,Y).\phi = 0$$

on M, for all $X, Y \in \chi(M)$.

Let M be a $(2n+1)$-dimensional ϕ- Quasi conformally semi-symmetric generalized Sasakian-space-form then $W(X,Y).\phi = 0$ reduces in to

$$(W(X,Y).\phi)Z = W(X,Y)\phi Z - \phi W(X,Y)Z = 0, \tag{39}$$

for any vector fields X, Y and $Z \in \chi(M)$.

Now, from (2) we have

$$W(X,Y)\phi Z = aR(X,Y)\phi Z + b[S(Y,\phi Z)X - S(X,\phi Z)Y + g(Y,\phi Z)QX - g(X,\phi Z)QY]$$
$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}[g(Y,\phi Z)X - g(X,\phi Z)Y]. \tag{40}$$

Using (1), (21) and (23) in (40), we have

$$\begin{aligned} W(X,Y)\phi Z = {} & af_2[-g(X,Z)\phi Y + \eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X \\ & -\eta(Y)\eta(Z)\phi X - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi] \\ & +\{af_3 + b(3f_2 + (2n-1)f_3)\}[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi] \\ & +\left\{af_1 + 2b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\} \\ & [g(Y,\phi Z)X - g(X,\phi Z)Y]. \end{aligned} \tag{41}$$

Further from (2), we have

$$\phi W(X,Y)Z = \phi aR(X,Y)Z + b[S(Y,Z)\phi X - S(X,Z)\phi Y + g(Y,Z)Q\phi X - g(X,Z)Q\phi Y]$$
$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}[g(Y,Z)\phi X - g(X,Z)\phi Y]. \tag{42}$$

Using (1), (21) and (23) in (42), we have

$$\phi W(X,Y)Z = af_2[-g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X - g(Y,\phi Z)\eta(X)\xi - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi] + \{af_3 + b(3f_2 + (2n-1)f_3)\}[\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X] + \left\{af_1 + 2b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\} [g(Y,Z)\phi X - g(X,Z)\phi Y]. \tag{43}$$

Substituting (41) and (43) in (39), we get

$$af_2[-g(X,Z)\phi Y + \eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi + g(X,\phi Z)Y - g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)X + g(Y,\phi Z)\eta(X)\xi + 2g(X,\phi Y)Z - 2g(X,\phi Y)\eta(Z)\xi] + \{af_3 + b(3f_2 + (2n-1)f_3)\} [g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi - \eta(X)\eta(Z)\phi Y + \eta(Y)\eta(Z)\phi X] + \left\{af_1 + 2b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\} [g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0. \tag{44}$$

Taking $Y = \xi$ in (44), we have

$$\{af_3 + b(3f_2 + (2n-1)f_3)\}[g(X,\phi Z)\xi + \eta(Z)\phi X] + \left\{af_1 + 2b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\} [-g(X,\phi Z)\xi - \eta(Z)\phi X] = 0. \tag{45}$$

Taking $Z = \xi$ in (45), we get

$$\left[af_3 + b(3f_2 + (2n-1)f_3) - \left\{af_1 + 2b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\}\right] \phi X = 0. \tag{46}$$

Since $\phi X \neq 0$, therefore from (46) we have

$$r = 2n(2n+1)\left[\frac{a(f_1 - f_3) + b(4nf_1 + 3f_2 - (2n+1)f_3)}{a + 2nb}\right].$$

This leads to the following result:

Theorem 4.1: A $(2n+1)$-dimensional generalized Sasakian-space-forms is ϕ-quasi-conformally semi-symmetric if $r = 2n(2n+1)\left[\frac{a(f_1-f_3)+b(4nf_1+3f_2-(2n+1)f_3)}{a+2nb}\right]$.

5. ξ −pseudo-projectively flat generalized Sasakian-space-forms

Definition 5.1: An almost contact metric manifold $(M^{2n+1}, g), n > 1$, is said to be ξ-pseudo-projectively flat if it satisfies the condition

$$\tilde{P}(X,Y)\xi = 0$$

on M, for all $X, Y \in \chi(M)$.

Let M be a $(2n+1)$-dimensional ξ-pseudo-projectively flat generalized Sasakian-space-forms then from (4), we have

$$aR(X,Y)\xi + b[S(Y,\xi)X - S(X,\xi)Y] - \frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}[g(Y,\xi)X - g(X,\xi)Y] = 0. \quad (47)$$

Using (18) and (22) in (47), we get

$$\left[a(f_1 - f_3) + 2nb(f_1 - f_3) - \frac{r}{(2n+1)}\left\{\frac{a}{2n} + b\right\}\right][\eta(Y)X - \eta(X)Y] = 0. \quad (48)$$

Since $[\eta(Y)X - \eta(X)Y] \neq 0$, then from (48), we have

$$\left[\frac{(a+2nb)\{2n(2n+1)(f_1-f_3)-r\}}{2n(2n+1)}\right] = 0. \quad (49)$$

Hence from (49), we have

Either $a + 2nb = 0$ or $r = 2n(2n+1)(f_1 - f_3)$. (50)

This leads to the following result:

Theorem 5.1: A $(2n+1)$-dimensional generalized Sasakian-space-forms is ξ-pseudo-projectively flat if either $a + 2nb = 0$ or $r = 2n(2n+1)(f_1 - f_3)$.

If $a + 2nb \neq 0$, then from (50) and (24), we get

$$f_3 = \frac{3f_2}{(1-2n)}. \quad (51)$$

This leads to the following result:

Corollary 5.1: A $(2n+1)$ dimensional generalized Sasakian-space-forms is ξ-pseudo-projectively flat if $f_3 = \frac{3f_2}{(1-2n)}$.

From (21) and (51), we have

$$S(X,Y) = 2n\left(f_1 - \frac{3f_2}{(1-2n)}\right)g(X,Y). \quad (52)$$

This leads to the following result:

Corollary 5.2: A ξ-pseudo-projectively flat generalized Sasakian-space-forms is an Einstein manifold.

6. ϕ-pseudo-projective semi-symmetric generalized Sasakian-space-forms

Definition 6.1: An almost contact metric manifold $(M^{2n+1}, g), n > 1$, is said to be ϕ-pseudo-projective semi-symmetric if it satisfies the condition

$$\tilde{P}(X,Y).\phi = 0.$$

on M, for all $X, Y \in \chi(M)$.

Let M be a $(2n+1)$-dimensional ϕ-pseudo-projective semi-symmetric generalized Sasakian-space-forms then $\tilde{P}(X,Y).\phi = 0$ reduces in to

$$\left(\tilde{P}(X,Y).\phi\right)Z = \tilde{P}(X,Y)\phi Z - \phi\tilde{P}(X,Y)Z = 0, \tag{53}$$

for any vector fields X, Y and $Z \in \chi(M)$.

Now, from (4) we have

$$\tilde{P}(X,Y)\phi Z = aR(X,Y)\phi Z + b[S(Y,\phi Z)X - S(X,\phi Z)Y]$$
$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}[g(Y,\phi Z)X - g(X,\phi Z)Y]\ . \tag{54}$$

Using (1) and (21) in (54), we have

$$\tilde{P}(X,Y)\phi Z = af_2[-g(X,Z)\phi Y + \eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X$$
$$-\eta(Y)\eta(Z)\phi X - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi]$$
$$+af_3[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi]$$
$$+\left\{af_1 + b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+b\right)\right\}$$
$$[g(Y,\phi Z)X - g(X,\phi Z)Y]. \tag{55}$$

Further, from (4), we have

$$\phi\tilde{P}(X,Y)Z = \phi aR(X,Y)Z + b[S(Y,Z)\phi X - S(X,Z)\phi Y]$$
$$-\frac{r}{(2n+1)}\left\{\frac{a}{2n}+b\right\}[g(Y,Z)\phi X - g(X,Z)\phi Y]. \tag{56}$$

Using (1) and (21) in (56), we have

$$\phi\tilde{P}(X,Y)Z = af_2[-g(X,\phi Z)Y + g(X,\phi Z)\eta(Y)\xi + g(Y,\phi Z)X$$
$$-g(Y,\phi Z)\eta(X)\xi - 2g(X,\phi Y)Z + 2g(X,\phi Y)\eta(Z)\xi]$$
$$+[af_3 + b(3f_2 + (2n-1)f_3]\{\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X\}$$
$$+\left\{af_1 + b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n}+b\right)\right\}$$
$$[g(Y,Z)\phi X - g(X,Z)\phi Y]. \tag{57}$$

Substituting (55) and (57) in (53), we get

$$af_2[-g(X,Z)\phi Y + \eta(Z)\eta(X)\phi Y + g(Y,Z)\phi X - \eta(Y)\eta(Z)\phi X - 2g(X,\phi Y)Z$$
$$+2g(X,\phi Y)\eta(Z)\xi + g(X,\phi Z)Y - g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)X + g(Y,\phi Z)\eta(X)\xi$$
$$+2g(X,\phi Y)Z - 2g(X,\phi Y)\eta(Z)\xi] + af_3[g(X,\phi Z)\eta(Y)\xi - g(Y,\phi Z)\eta(X)\xi$$
$$-\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X] - b(3f_2 + (2n-1)f_3)[\eta(X)\eta(Z)\phi Y - \eta(Y)\eta(Z)\phi X]$$

$$+\left\{af_1 + b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\}$$

$$[g(Y,\phi Z)X - g(X,\phi Z)Y - g(Y,Z)\phi X + g(X,Z)\phi Y] = 0. \tag{58}$$

Taking $Y = \xi$ in (58), we have

$$af_3[\, g(X,\phi Z)\xi + \eta(Z)\phi X] - b(3f_2 + (2n-1)f_3)[-\eta(Z)\phi X]$$

$$+\left\{af_1 + b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\}[-g(X,\phi Z)\xi - \eta(Z)\phi X] = 0. \tag{59}$$

Taking $Z = \xi$, in (59), we have

$$\left[af_3 + b(3f_2 + (2n-1)f_3) - \left\{af_1 + b(2nf_1 + 3f_2 - f_3) - \frac{r}{(2n+1)}\left(\frac{a}{2n} + b\right)\right\}\right]\phi X = 0. \tag{60}$$

Since $\phi X \neq 0$, therefore from (60) we have

$$\left[\frac{(a + 2nb)\{2n(2n+1)(f_1 - f_3) - r\}}{2n(2n+1)}\right] = 0.$$

$\Rightarrow$ either $a + 2nb = 0$ or $r = 2n(2n+1)(f_1 - f_3)$.

This leads to the following result:

Theorem 6.1: A $(2n+1)$- dimensional generalized Sasakian-space-forms is ϕ-projectively semi-symmetric if either $a + 2nb = 0$ or $r = 2n(2n+1)(f_1 - f_3)$.

7. $\mathbb{Q}$-curvature pseudo-symmetric generalized Sasakian-space-forms

Let M be a $(2n+1)$-dimensional $\mathbb{Q}$-curvature pseudo-symmetric generalized Sasakian-space-forms then from (3), we have

$$(R(\xi,Y).\mathbb{Q})(U,V)W = L_{\mathbb{Q}}\big[\big((\xi\wedge Y)\mathbb{Q}\big)(U,V)W\big]. \tag{61}$$

Equation (61) yields

$$R(\xi,Y)\mathbb{Q}(U,V)W - \mathbb{Q}(R(\xi,Y)U,V)W - \mathbb{Q}(U,R(\xi,Y)V)W - \mathbb{Q}(U,V)R(\xi,Y)W$$
$$= L_{\mathbb{Q}}[(\xi\wedge_S Y)\mathbb{Q}(U,V)W - \mathbb{Q}\big((\xi\wedge_S Y)U,V\big)W - \mathbb{Q}(U,(\xi\wedge_S Y)V)W$$
$$-\mathbb{Q}(U,V)(\xi\wedge_S Y)W]. \tag{62}$$

Using (8) and (19) in (62), we get

$$\big((f_1 - f_3) - L_{\mathbb{Q}}\big)[g(\mathbb{Q}(U,V)W,Y)\xi - \eta(\mathbb{Q}(U,V)W)Y - \mathbb{Q}(g(Y,U)\xi - \eta(U)Y,V)W$$
$$-\mathbb{Q}(U,g(Y,V)\xi - \eta(V)Y)W - \mathbb{Q}(U,V)(g(Y,W)\xi - \eta(W)Y)] = 0. \tag{63}$$

Taking inner product with ξ, we obtain

$$\big((f_1 - f_3) - L_{\mathbb{Q}}\big)[\mathbb{Q}(U,V,W,Y) - \eta(Y)\eta(\mathbb{Q}(U,V)W) + \eta(U)\eta(\mathbb{Q}(Y,V)W)$$
$$+\eta(V)\eta(\mathbb{Q}(U,Y)W) + \eta(W)\eta(\mathbb{Q}(U,V)Y) - g(Y,U)\eta(\mathbb{Q}(\xi,V)W)$$

$-g(Y,V)\eta(\mathbb{Q}(U,\xi)W) - g(Y,W)\eta(\mathbb{Q}(U,V)\xi)] = 0.$ (64)

Putting $U = Y$ in (64), we get either $L_{\mathbb{Q}} = (f_1 - f_3)$ or,

$[\mathbb{Q}(Y,V,W,Y) + \eta(V)\eta(\mathbb{Q}(Y,Y)W) + \eta(W)\eta(\mathbb{Q}(Y,V)Y) - g(Y,Y)\eta(\mathbb{Q}(\xi,V)W)$

$-g(Y,V)\eta(\mathbb{Q}(Y,\xi)W) - g(Y,W)\eta(\mathbb{Q}(Y,V)\xi)] = 0.$ (65)

On contracting (65) w.r.to $Y$and using (1), (3), (14), (16),(21) and (31), we obtain

$S(V,W) = Ag(V,W) + B\eta(V)\eta(W).$ (66)

where $A = 2n\left[f_1 - f_3 - \frac{\Psi}{2n}\right]$ and $B = -4n\left[f_1 - f_3 - \frac{\Psi}{2n}\right]$.

This leads to the following result:

Theorem 7.1. If a $(2n+1)$-dimensional generalized Sasakian-space-forms is $\mathbb{Q}$-curvature pseudo-symmetric then either $L_{\mathbb{Q}} = (f_1 - f_3)$ or the manifold is η-Einstein.

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