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ON HOMOGENEOUS GENERALIZED m-KROPINA SPACES WITH CURVATURE ASPECTS

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Abstract: The motive of this article is to calculate the explicit formula for S-curvature and mean Berwald curvature in homogeneous generalized m-Kropina metric. We establish the necessary and sufficient condition for the generalized m-Kropina metric to be Riemannian or locally Minkowskian.

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1. Introduction

It is stated that inner product spaces of every dimension up to isomorphism are known to be unique in Riemannian geometry, however, this is not the case for the Minkowski norm. Finsler geometry includes Riemannian, Euclidean, and Minkowskian geometries as special cases hence, allowing it to describe a wide range of physics and continuum mechanics phenomena. In the continuum mechanical behavior of solids, the importance of Finsler geometry was observed by many physicists [8, 13]. Along with mechanics, the importance of Finsler geometry in describing fundamental descriptions of other branches of physics such as electromagnetism, quantum theory, and gravitation is striking. As we know that various Finsler metrics on the tangent space of a Finsler manifold are not isomorphic.

In Shen's words, a Finsler manifold is quite colorful. Thus, it is very fascinating to study Finsler manifolds with a single color. Homogeneous Finsler spaces are examples of Finsler manifolds with a single color, which have been studied by Deng [11] Latifi, and Razavi [14]. The origin of homogenous Riemannian spaces was initiated with the Myers-Steenrod theorem in [1939] which states that a group of isometries of a connected Riemannian manifold admits a differentiable structure such that it forms a Lie transformation group of the manifold. This theorem was a breakthrough since it extended the scope of applying Lie theory to all homogeneous Riemannian manifolds. Homogeneous spaces are a natural generalization of symmetric spaces and these spaces retain many of their great properties. One of the important properties is the existence of a transitive group of transformations, which are sometimes called symmetries.

In Finsler geometry, S-curvature is a significant geometric measure because of its ambiguous connections to other quantities like flag curvature, the Ricci scalar, and others. Shen proved that for Finsler spaces with vanishing S-curvature, the Bishop-Gromov volume comparison theorem is valid. As a result, it is important to know homogeneous Finsler spaces as possessing vanishing S-curvature.

Deng [11] and Shen [22] examined the non-Riemannian quantities such as Cartan torsion, Landsberg curvature, mean Landsberg curvature, Berwald curvature, S-curvature which disappear in the case of Riemannian [23]. While studying curvature properties of homogeneous spaces by using the formula for the S-curvature of a left-invariant Riemannian metric on a Lie group, Milnor [15] calculated S-curvature in homogeneous spaces. In [1, 9] authors have studied the construction and algebraic description of invariant (α,β)-metrics on homogeneous spaces. Geometrically, S-curvature is the study of the rate of change of distortions along geodesics. It is known that S-curvature is a non-Riemannian quantity means every Riemannian manifold has to vanish S-curvature. The notion of S-curvature was introduced by Shen [20] for given comparison theorems on Finsler manifolds in 1997. This non-Riemannian quantity is used for the characterization of Finsler metrics among Berwald metric, Riemannian metric, and Locally Minkowskian metric. Shen [7] also gave an explicit formula for S-curvature in a local coordinate system.

However, in a homogeneous Finsler space, geometers were interested in finding the S-curvature formula irrespective of the local coordinate system. For homogeneous Randers space, the S-curvature formula was calculated by Deng [10]. Further, the calculation of S-curvature on homogeneous (α,β)-metrics was another task completed by Deng and Wang [12]. Recently, Shanker and Kaur rectified the S-curvature formula for homogeneous (α,β)-metric provided by Deng and Wang [12]. Later, the calculation of S-curvature on a homogeneous Finsler space with square metric and Randers changed square metric have been accomplished in [17, 19]. The purpose of this paper is to calculate S-curvature and mean Berwald curvature in homogeneous generalized m-Kropina Finsler spaces. Along with this we also prove the following result:

Theorem 1.1 *Let F be generalized m-Kropina metric with scalar flag curvature $K = K(x,y)$ on a Finsler space. Then, F is weak Berwald metric if and only if F is Berwald metric and $K = 0$. Then, F must be locally Minkowskian.*

2. Preliminaries

In this section, we discuss basic definitions and notations of Finsler geometry. For more elaborate concepts of Finsler geometry and homogeneous Finsler geometry, refer [2, 7, 11, 21]. Let V be an n-dimensional real vector space endowed with smooth norm F on $V\backslash\{0\}$, which is non-negative, i.e., $F(u) \geq 0 \forall\ u\ \in V$, positively homogeneous, i.e., $F(\lambda u) = \lambda F(u)\ \forall\ \lambda > 0$, and strongly convex, i.e., if $\{u_1, u_2, \ldots, u_n\}$ be the basis of V

such that $y = y^1 u_1 + y^2 u_2 + \ldots + y^n u_n$, then the Hessian matrix $(g_{ij}) := \left(\left[\frac{1}{2}F^2\right]_{y^i y^j}\right)$, is positive definite at every point of $V\backslash\{0\}$. The pair (V, F) is called Minkowski space and F is called Minkowski norm.

Let M be a connected (smooth) manifold. A Finsler metric on M is a function $F: TM \to [0, \infty)$ which satisfies:

1. F is smooth on slit tangent bundle $TM\backslash\{0\}$,

2. The restriction of F to any $T_x M, x \in M$ is a Minkowski norm.

The space (M, F) is called Finsler space. Let $\gamma: [0,1] \to M$ be a C^1-curve. Then Finsler length $L(\gamma)$ of γ is defined as

$L(\gamma) = \int_0^1 F(\gamma(t), \gamma'(t))dt.$

Further, Finsler distance $d_F(p, q)$ between two points $p, q \in M$ is defined as

$d_F(p, q) = inf_\gamma L(\gamma),$

where infimum is taken over all piecewise C^1-curves joining p and q. In this paper, we use the concept of Berwald connection where horizontal and derivative is denoted by ``|". And horizontal covariant derivatives of I, mean Cartan torsion along with geodesics gives mean Landsberg curvature $J_y(u) := J_i(y)u^i$, where $J_i := I_{i|s} y^s$.

Definition 2.1 *Let $F = \alpha\phi(s); s = \beta/\alpha$, where ϕ is a smooth function on an open interval $(-b_0, b_0), \alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold with $||\beta|| < b_0$. Then, F is Finsler metric if and only if following conditions are satisfied:*

$$\phi(s) > 0,\ \phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0\ \ \forall\ |s| \leq b < b_0. \tag{1}$$

In this, if ϕ does not satisfy the equation 1 or $\phi(0)$ is not defined, then (α, β)-metric is said to be singular Finsler metric.

Let (M, F) be a Finsler space. A diffeomorphism of M onto itself is said to be isometry, if it preserves the Finsler function, i.e., $F(\phi(p), d\phi_p(X)) = F(p, X)$ for any $p \in M$ and $X \in T_p M$. Let G be a Lie group and M a smooth manifold. If G has smooth action on M, then G is called Lie transformation group of M. A connected Finsler space (M, F) is said to be homogeneous Finsler space, if action of group of isometries of (M, F), denoted by $I(M, F)$ is transitive on M.

Let $G \subset I(M, F)$ be a connected Lie group acting transitively on Finsler space (M, F), and at a fixed point $p \in M$, let H be its isotropy group. Then M can be written as coset space G/H, with a G-invariant Finsler metric F. It is evident to see that H is comapct, since action of H leaves invariant unit sphere in $T_p M$. Hence, we obtain reductive decomposition of $\mathfrak{g}$, Lie algebra of G as

$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$,

where $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras of G and H respectively and $\mathfrak{m} \subset \mathfrak{g}$ is a vector subspace such that $Ad(\mathfrak{h})(\mathfrak{m}) \subset \mathfrak{m}$, $\forall h \in H$, where Ad denotes Adjoint representation of G.

Remark 1 *A homogeneous Finsler manifold $M = G/H$ is reductive homogeneous space.*

Next proposition shows that G-invariant Finsler metrics on G/H can be identified with Minkowski norm F as follows:

Proposition 2.1 *[9] Let G/H be a reductive homogeneous manifold satisfying $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then there exists a one-to-one correspondence between the G-invariant Finsler metrics on G/H and the Minkowski norms F on $\mathfrak{m}$ which satisfy*

$$F(Ad(h))(x) = F(x), \qquad \forall\ h \in H, x \in \mathfrak{m}.$$

On considering $\phi(s) = \frac{1}{s^m}, (s \neq 0)$, we get an important class of (α, β)- metrics known as generalized m-Kropina metric. Hence, generalized m-Kropina metric is defined as $F = \frac{\alpha^{m+1}}{\beta^m}$.

Generalized m-Kropina metric can also be expressed using the following Lemma:

Lemma 2.1 Let (M, α) be a Riemannian space. Then the generalized m-Kropina space, (M, F) where $F = \frac{\alpha^{m+1}}{\beta^m}, (m \neq -1,0,1)$ $\beta = b_i y^i$, a 1-form with $||\beta|| = \sqrt{b_i b^i} < 1$, consists of Riemannian metric α along with a smooth vector field X on M, which satisfies $\alpha(X|_x) < 1\ \forall\ x \in M$, i.e.,

$$F(x, y) = \frac{\alpha(x, y)^{m+1}}{\langle X|_x, y\rangle^m}.$$

Consider an n-dimensional vector space V, with F be Minkowski norm on V. For $\{e_i\}$ to be the basis of V, define a quantity as

$$\sigma_F = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(y^i e^i) < 1\}},$$

where Vol denotes Volume of a subset in Euclidean space $\mathbb{R}^n$ and B^n is the open ball with radius of 1. Generally, this quantity is dependent on basis $\{e_i\}$. Next, we recall the definition of S-curvature in Finsler spaces:

Definition 2.2 *The distortion of an n-dimensional vector space is defined as*

$$\tau(y) = ln \frac{\sqrt{det(g_{ij}(y))}}{\sigma_F}.$$

For $x \in M$, let $\tau(x, y)$ be distortion of Minkowski norm F_x on $T_x M$, $x \in M$. Let $\gamma(t)$ be a geodesic satisfying $\gamma(0) = x$ & $\dot{\gamma}(0) = y$. Then rate of change of distortion along geodesic γ is said to be S-curvature and is defined as

$S(x,y) = \frac{d}{dt}[\tau(\gamma(t)), \dot{\gamma}(t)]|_{t=0}$.

Definition 2.3 *A Finsler space $(G/H, F)$ is said to have almost isotropic S-curvature, if there exists a smooth function $d(x)$ on M and closed 1-form η which satisfies:*

$$S(x,y) = (n+1)(d(x)F(y) + \eta(y)), \qquad x \in G/H, \qquad y \in T_x M.$$

If $\eta(y) = 0$, then $(G/H, F)$ is said to have isotropic S-curvature.

And if $\eta(y) = 0, c(x) =$ constant, then $(G/H, F)$ is said to have constant S-curvature.

Definition 2.4 *The Busemann-Hausdroff volume form, $dV_{BH} = \sigma_{BH}(x)dx$ is defined as:*

$$\sigma_{BH} = \frac{\mathrm{Vol}(B^n)}{\mathrm{Vol}\{(y^i) \in \mathbb{R}^n | F(x, y^i e^i) < 1\}}.$$

The Holmes-Thompson volume form, $dV_{HT} = \sigma_{HT}(x)dx$ is defined as :

$$\sigma_{HT}(x) = \frac{1}{\mathrm{Vol}B^n}\int_{\{y^i \in \mathbb{R}^n | F(x, y^i e^i) < 1\}} det(g_{ij})dy.$$

Both volume forms coincide in the case of Riemannian metric, i.e., $dV_{BH} = dV_{HT} = \sqrt{det g_{ij}(x)dx}.$

3. S-Curvature and Mean Berwald Curvature

3.1 S-curvature of homogeneous generalized m-Kropina space

Shen and Cheng [5] in 2009, discovered the formula for S-curvature of an (α, β)-metric in local coordinate system, which is given as follows:

$$S = \left(2\psi - \frac{f'(b)}{bf(b)}\right)(r_0 + s_0) - \alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00} - 2\alpha Q s_0), \text{ where} \tag{2}$$

$$Q = \frac{\phi'}{\phi - s\phi'}, \qquad \psi = \frac{Q'}{2\Delta},$$

$$\Delta = 1 + sQ + (b^2 - s^2)Q',$$

$$\Phi = (sQ' - Q)(1 + n\Delta + sQ) - (b^2 - s^2)(1 + sQ)Q'',$$

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \qquad r_j = b^i r_{ij}, \qquad r_0 = r_i y^i, \qquad r_{00} = r_{ij} y^i y^j,$$

$$s_{ij} := \frac{1}{2}(b_{i|j} - b_{j|i}), \qquad s_j = b^i s_{ij}, \qquad s_0 = s_i y^i,$$

And $f(b)$ used in equation 2 is defined as

$$f(b)=\begin{cases}\dfrac{\int_0^\pi sin^{n-2}tdt}{\int_0^\pi \frac{sin^{n-2}t}{\phi(bcost)^n}dt} & \text{if } dV=dV_{BH},\\[2ex] \dfrac{\int_0^\pi (sin^{n-2}t)T(bcost)dt}{\int_0^\pi (sin^{n-2}t)dt} & \text{if } dV=dV_{HT}.\end{cases}$$

where, dV_{BH}, dV_{HT} are Busemann-Hausdorff volume form and Holmes-Thompson volume form are defined in defintion 2.4.

We can easily see that in the case of constant Riemannian length b, the parameter r_0+s_0 vanishes, hence equation 2 reduces to

$$S=\alpha^{-1}\frac{\Phi}{2\Delta^2}(r_{00}-2\alpha Qs_0)$$

Next, we recall the amendment done in S-curvature formula in homogeneous Finsler spaces proved in [18].

Theorem 3.1 *[18] Let $F=\alpha\phi(s)$ be a G-invariant (α,β)- metric on the reductive homogeneous Finsler space G/H with a decomposition of the Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$. Then S-curvature is given by*

$$S(H,y)=\frac{\Phi}{2\alpha\Delta^2}(\langle[v,y]_{\mathfrak{m}},y\rangle+\alpha Q\langle[v,y]_{\mathfrak{m}},v\rangle), \tag{3}$$

where, $v\in\mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Using Theorem 3.1, we deduce formula for S-curvature in the homogeneous generalized m-Kropina spaces.

Theorem 3.2 *Let G/H be a reductive homogeneous Finsler spaces with decomposition of Lie algebra $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, and $F=\frac{\alpha^{m+1}}{\beta^m}$ be a G-invariant generalized m-Kropina metric. Then S-curvature is given by*

$$\frac{ms[(n-nm)s^2+(nm+1)b^2]}{[(1-m)s^2+b^2m]^2}[\frac{1}{\alpha}\langle[v,y]_{\mathfrak{m}},y\rangle-\frac{m}{(m+1)s}\langle[v,y]_{\mathfrak{m}},v\rangle], \tag{4}$$

where, $v\in\mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. In order to calculate S-curvature in reductive homogeneous Finsler space with G-invariant generalized m-Kropina metric, we first find the parameters to be used in modified version of S-curvature in Theorem 3.1. Let us recall the following parameters:

$$Q=\frac{\phi'}{\phi-s\phi'}, \qquad \Delta=1+sQ+(b^2-s^2)Q',$$

$\Phi = -(Q - sQ')(1 + n\Delta + sQ) - (b^2 - s^2)(1 + sQ)Q''.$

By direct computations we get

$$Q = \frac{-m}{s(1+m)}, \quad \Delta = \frac{(1-m)s^2 + b^2m}{(1+m)s^2}, \Phi = \frac{2m}{(1+m)^2s^3}[(n-nm)s^2 + (nm+1)b^2].$$

Now, we substitute all these values in equation 3, which gives

$$S(H,y) = \frac{ms[(n-nm)s^2 + (nm+1)b^2]}{[(1-m)s^2 + b^2m]^2}[\frac{1}{\alpha}\langle[v,y]_{\mathfrak{m}}, y\rangle - \frac{m}{(m+1)s}\langle[v,y]_{\mathfrak{m}}, v\rangle]$$

where, $v \in \mathfrak{m}$ corresponds to the 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H. This proves Theorem 3.2.

A straightforward application of Theorem 3.2 can be seen as:

Corollary 3.1 *Let $(G/H,F)$ be a homogeneous generalized m-Kropina Finlser space with similar presumptions as taken in Theorem 3.2. Then homogeneous generalized m-Kropina metric has isotropic S-curvature if and only if S-curvature of F equal to zero.*

Proof. Using the definition 2.3, we recall that if $(G/H,F)$ has isotropic S-curvature implies

$$S(x,y) = (n+1)(d(x)F(y) + \eta(y)), \qquad x \in G/H, \qquad y \in T_xM.$$

Since in the case of homogeneous space, it is enough to calculate S-curvature at origin, i.e., $x = H$, $y = v$. Put $x = H$, $y = v$ in the formula of S- curvature deduced in Theorem 3.2, we get $d(H) = 0$, which implies $S(H,y) = 0$ $\forall y \in T_xM$. Hence, G/H has zero S-curvature.

3.2 Mean Berwald curvature of homogeneous generalized m-Kropina spaces

Next, we emphasises on calculating mean Berwald curvature for homogeneous Finsler spaces with generalized m-Kropina metric.

We first recall the concept of mean Berwald curvature in homogeneous Finsler spaces. In [7], author discussed the notion of mean Berwald curvature for Finsler spaces as follows:

Consider $E_{ij} = \frac{1}{2}\frac{\partial^2 S(x,y)}{\partial y^i \partial y^j} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}\left(\frac{\partial G^m}{\partial y^m}\right)(x,y)$, where G^m are geodesic spray coefficients.

Consider the symmetric forms $E_y: T_xM \times T_xM \to \mathbb{R}$ expressed as

$$E_y(a,b) = E_{ij}a^ib^j, \text{where } a = a^i\frac{\partial}{\partial y^i}, b = b^j\frac{\partial}{\partial y^j} \in T_xM, \qquad x \in M.$$

Then E-tensor defined as family of symmetric forms $\{E_y\}$ can be seen as:

$E = \{E_y : y \in TM/\{0\}\},$

which is known as E-Curvature or mean Berwald curvature.

Before moving to main result of this section let us find some important quantities to be used further:

We know that at origin,

$$a_{ij} = \delta_i^j,\ y_i = y^i,\ \alpha_{y_i} = \frac{y_i}{\alpha},\ \beta_{y^i} = b_i,\ \ s_{y^i} = \frac{\alpha b_i - s y_i}{\alpha^2},$$

$$s_{y_i y_j} = \frac{3 s y_i y_j - (b_i y_j + b_j y_i)\alpha - \alpha^2 s \delta_j^i}{\alpha^4}.$$

For the sake of simplicity, we denote $\Omega = \frac{mn(1-m)s^3 + m(nm+1)b^2 s}{[(1-m)s^2 + b^2 m]^2}$ and find the following quantities.

$$\frac{\partial \Omega}{\partial y^j} = \frac{1}{[(1-m)s^2+b^2m]^3}[3mn(1-m)^2 s^4 + m(1-m)(1+nm+3mnb^2)s^2 \\ -4mn(1-m)s^3 + m(nm+1)b^2 s + m^2(nm+1)b^4] s_{y^j}, \tag{5}$$

$$\frac{\partial^2 \Omega}{\partial y^i \partial y^j} = \left[\frac{-6mn(1-m)^3 s^5 + 12mn(1-m)^2 s^4 - 4m(1-m)^2(1+nm)s^3 - m(1-m)(5+17mn)b^2 s^2 + 2m^2(1-m)(mn-2)b^2 s + m^2(1+mn)b^4}{[(1-m)s^2+b^2m]^4}\right] s_{y^j} s_{y^i} \\ + \left[\frac{\{3mn(1-m)^2 s^4 + m(1-m)(1+nm+3mnb^2)s^2 - 4mn(1-m)s^3 + m(nm+1)b^2 s + m^2(nm+1)b^4\} s_{y^j}}{[(1-m)s^2+b^2m]^3}\right] s_{y^i y^j}. \tag{6}$$

Now, we are ready to prove our next Theorem:

Theorem 3.3 *Let $(G/H, F)$ be a reductive homogeneous Finsler spaces with decomposition of Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, and $F = \frac{\alpha^{m+1}}{\beta^m}$ be a G-invariant generalized m- Kropina metric. Then mean Berwald curvature is given by*

$$\begin{aligned} E_{i,j}(H,y) = \ & \frac{1}{2}[(\frac{1}{\alpha}\Omega_{ij} - \frac{y_i}{\alpha^3}\Omega_j - \frac{y^j}{\alpha^3}\Omega_i - \frac{A}{\alpha^3}\delta_i^j + \frac{3\Omega}{\alpha^5}y^i y^j)\langle [v,y]_{\mathfrak{m}}, y\rangle \\ & + (\frac{1}{\alpha}\Omega_j - \frac{\Omega y_j}{\alpha^3})(\langle [v,v_i]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_i\rangle) \\ & + (\frac{1}{\alpha}\Omega_i - \frac{\Omega y_i}{\alpha^3})(\langle [v,v_j]_{\mathfrak{m}}, y\rangle + \langle [v,y]_{\mathfrak{m}}, v_j\rangle) \\ & + \frac{\Omega}{\alpha}(\langle [v,v_j]_{\mathfrak{m}}, vi\rangle + \langle [v,v_i]_{\mathfrak{m}}, v_j\rangle) + (\frac{-m}{m+1}\frac{\Omega_i}{s^2}s_{y^j} + \frac{2m}{(m+1)s^3}\Omega s_{y^i}s_{y^j} \\ & - \frac{m}{(m+1)s^2}\Omega s_{y^j y^i} + \frac{m}{(m+1)s}\Omega_{ji} - \frac{m}{(m+1)s^2}s^{y^i}\Omega_j)\langle [v,y]_{\mathfrak{m}}, v\rangle \\ & + (\frac{-m\Omega}{(m+1)s^2}s_{y^j} + \frac{m}{(m+1)s}\Omega_j)\langle [v,v_i]_{\mathfrak{m}}, v\rangle] \end{aligned} \tag{7}$$

where, $\Omega = \frac{ms[(n-nm)s^2 + (nm+1)b^2]}{[(1-m)s^2+b^2m]^2}$, also $v \in \mathfrak{m}$ corresponds to 1-form β and $\mathfrak{m}$ is identified with the tangent space $T_H(G/H)$ of G/H at the origin H.

Proof. From Theorem 3.1, recall that S-curvature at origin given by equation 4 is as follows:

$$S(H,y) = \frac{ms[(n-nm)s^2+(nm+1)b^2]}{[(1-m)s^2+b^2m]^2}[\frac{1}{\alpha}\langle [v,y]_{\mathfrak{m}}, y\rangle - \frac{m}{(m+1)s}\langle [v,y]_{\mathfrak{m}}, v\rangle].$$

Consider $S(H,y) = A + B$, where $A = \frac{\Omega}{\alpha}\langle [v,y]_m, y\rangle$ and $B = \frac{-m\Omega}{(m+1)s}\langle [v,y]_m, v\rangle$.

Now, by definition of mean Berwald curvature, we know that

$$E_{ij} = \frac{1}{2}\frac{\partial^2 S(H,y)}{\partial y^i \partial y^j} = \frac{1}{2}\left(\frac{\partial^2 A}{\partial y^i y^j} + \frac{\partial^2 B}{\partial y^i y^j}\right). \tag{8}$$

Next, we calculate $\frac{\partial^2 A}{\partial y^i y^j}$ and $\frac{\partial^2 B}{\partial y^i y^j}$ as follows:

$$\frac{\partial A}{\partial y^j} = \left(\frac{1}{\alpha}\frac{\partial \Omega}{\partial y^j} - \frac{\Omega}{\alpha^2}\frac{y^j}{\alpha}\right)\langle [v,y]_m, y\rangle + \frac{\Omega}{\alpha}\left(\langle [v,v_j]_m\rangle + \langle [v,y]_m, v_j\rangle\right),$$

$$\begin{aligned}\frac{\partial^2 A}{\partial y^i y^j} = \;& \left(\frac{1}{\alpha}\Omega_{ij} - \frac{y_i}{\alpha^3}\Omega_j - \frac{y^j}{\alpha^3}\Omega_i - \frac{A}{\alpha^3}\delta_i^j + \frac{3\Omega}{\alpha^5}y^i y^j\right)\langle [v,y]_m, y\rangle \\ & + \left(\frac{1}{\alpha}\Omega_j - \frac{\Omega y_j}{\alpha^3}\right)\left(\langle [v,v_i]_m, y\rangle + \langle [v,y]_m, v_i\rangle\right) \\ & + \left(\frac{1}{\alpha}\Omega_i - \frac{\Omega y_i}{\alpha^3}\right)\left(\langle [v,v_j]_m, y\rangle + \langle [v,y]_m, v_j\rangle\right) + \frac{\Omega}{\alpha}\left(\langle [v,v_j]_m, vi\rangle + \langle [v,v_i]_m, v_j\rangle\right)\end{aligned} \tag{9}$$

$$\frac{\partial B}{\partial y^j} = \left[\frac{-m\Omega}{(m+1)s^2}s_{y^j} + \frac{m}{(m+1)s}\Omega_j\right]\langle [v,y]_m, v\rangle + \left[\frac{m\Omega}{(m+1)s}\right]\langle [v,v_j]_m, v\rangle$$

$$\begin{aligned}\frac{\partial^2 B}{\partial y^i y^j} = \;& \left[\frac{-m}{m+1}\frac{\Omega_i}{s^2}s_{y^j} + \frac{2m}{(m+1)s^3}\Omega s_{y^i}s_{y^j} - \frac{m}{(m+1)s^2}\Omega s_{y^j y^i} + \frac{m}{(m+1)s}\Omega_{ji}\right. \\ & \left. - \frac{m}{(m+1)s^2}s^{y^i}\Omega_j\right]\langle [v,y]_m, v\rangle + \left[\frac{-m\Omega}{(m+1)s^2}s_{y^j} + \frac{m}{(m+1)s}\Omega_j\right]\langle [v,v_i]_m, v\rangle.\end{aligned} \tag{10}$$

Using equation 9 and equation 10 in equation 8, we get equation 7. Hence proved.

4. Necessary and sufficient condition for metric to be Riemannian or locally Minkowskian

In Finsler geometry, it is of great importance to find if a Finsler manifold can be isometrically immersed into Minkowskian space. Hence, geometers crave to find necessary and sufficient conditions for this Finsler space to be Minkowskian or locally Minkowskian. For example, in [3], authors have proved that conformally flat weakly Einstein (α,β)-metrics are either Riemannian or locally Minkowskian. Similarly, in [24], it is proved that a Berwald Finsler metric with scalar flag curvature must be locally Minkowskian. With this motivation, next we are going to establish some conditions for generalized m-Kropina metrics. In order to prove Theorem 1.1, we use the following Lemma.

Lemma 4.1 *[6] An (α,β)-metric is Riemannian metric if and only if $\Phi = 0$.*

4.1 Proof of Theorem 1.1

Proof. Using Lemma 4.1 and value of Φ, we get that F can't be Riemannian. Suppose F is of constant curvature K, then from [22], we know that J_i, Landsberg curvature satisfies the following equality:

$$J_{i|m}y^m + KF^2 I_i = 0,$$

where

$$J_i = \frac{-1}{2\alpha^4\Delta}\Big(\frac{2\alpha^2}{b^2-s^2}\Big[\frac{\Phi}{\Delta}+(n+1)(Q-sQ')\Big](r_0+s_0)h_i + \frac{\alpha}{b^2-s^2}\Big[\Psi_1+s\frac{\phi}{\Delta}\Big](r_{00}-2\alpha Qs_0)h_i + \alpha[-\alpha Q's_0h_i+\alpha Q(\alpha^2 s_i-\bar{y}_i s_0) + \alpha^2\Delta s_{i0}+\alpha^2(r_{i0}-2\alpha Qs_0)-(r_{00}-2\alpha Qs_0)\bar{y}_i]\frac{\Phi}{\Delta}\Big). \quad (11)$$

In [4], the following has been calculated:

$$J := J_i b^i = \frac{-1}{2\Delta\alpha^2}\{\Psi_1(r_{00}-2\alpha Qs_0)+\alpha\Psi_2(r_0+s_0)\},$$

where Ψ_1 and Ψ_2 are expressed in [4]

$$\Psi_1 = \sqrt{b^2-s^2}\,\Delta^{\frac{1}{2}}\left[\frac{\sqrt{b^2-s^2}}{\Delta^{\frac{3}{2}}}\right]', \quad \Psi_2 = 2(n+1)(Q-sQ')+\frac{3\Phi}{\Delta}.$$

We use the similar approach as in [16], and get that for F with constant flag curvature K, we have

$$J_{|m}y^m - J_l\frac{\partial(G^l-\bar{G}^l)}{\partial y^i} - 2\frac{\partial J_i}{\partial y^i}(G^l-\bar{G}^l)+K\alpha^2\phi^2 I_i = 0. \quad (12)$$

From [24], we get the

$$\frac{\Phi s_{i0}}{2\Delta\alpha}a^{ik}s_{k0} + \frac{\Phi s_{l0}}{2\Delta\alpha}(sQs_0^l + Q's_0^l(b^2-s^2)) - KF\frac{\Phi s_{i0}}{2\Delta}(\phi-s\phi')(b^2-s^2) = 0.$$

For assumption $\Phi \neq 0$, above equation can be deduced to

$$s_{i0}s_0^i\Delta - K\alpha^2\phi(\phi-s\phi')(b^2-s^2) = 0. \quad (13)$$

Using values for generalized m-Kropina metric, equation 13 can be written as

$$s_{i0}s_0^i\left[\frac{(1-m)\beta^2+b^2m\alpha^2}{(1+m)\beta^2}\right] - \frac{K[\alpha^2(1+m)(b^2-s^2)]}{s^{2m}} = 0. \quad (14)$$

which on simplification gives

$$s_{i0}s_0^i[(1-m)\beta^2+b^2m\alpha^2]\beta^{2m-2} = K(1+m)^2\alpha^{2m}(\alpha^2b^2-\beta^2), \quad (15)$$

which implies

$$s_{i0}s_0^i(1-m)\beta^{2m} + s_{i0}s_0^ib^2m\alpha^2\beta^{2m-2} + K(1+m)^2\alpha^{2m}\beta^2 = K(1+m)^2\alpha^{2m+2}b^2. \quad (16)$$

As we see that left term of the equation 16 is divisible by β^2, but right hand term is not. So we can get that flag curvature K vanishes, since $m \neq -1$. On plugging $K=0$ in equation 16, we obtain $s_{i0}s_0^i = a_{ij}(x)s_0^js_0^i = 0$. But since a_{ij} is positive definite, we get $s_0^i = 0$, which means β is closed. Also, by $r_{00}=0$ and $s_0=0$, we know that β is parallel with respect to α. Hence, it is proved that F is a Berwald metric and must be locally Minkoswkian.

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