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THE BICOMPLEX LAPLACE TRANSFORM OF RIEMANN-LIOUVILLE FRACTIONAL OPERATORS: PROPERTIES AND IMPLICATION

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Abstract: This article discusses Riemann-Liouville integrals and differentials of bicomplex order in terms of bicomplex Laplace transform. Analyzing bicomplex Laplace transforms of mixed type operators (composition of bicomplex order and integer order integral and differential operators), we obtain several important results. By using the obtained results, we obtain the solution of an initial value problem for a non-homogeneous fractional differential equation equipped with the Riemann-Liouville differential operator of bicomplex order.

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1. Introduction

The theory of bicomplex numbers has been the subject of active research for a long time since the fundamental work and discovery of this special algebra. The algebra of bicomplex numbers is widely used in the literature because it becomes a viable commutative alternative to the non-skewed field of quaternions introduced by Hamilton, since both are four-dimensional generalization of complex numbers. Various integral transforms are indispensable tools for discussing solutions of integer as well as fractional order differential equations viz. Laplace transform, double Laplace transform, triple Laplace transform, Laplace-Stieltjes transform, Fourier, Fourier-Stieltjes transform, Hankel transform, and Mellin transform etc.

A bicomplex extension of these integral transforms along with their various important properties and applications were discussed in [1-6, 16, 17, 24]. In the past decades, fractional calculus has become the focus of many research studies. Its value is not limited to just an interesting mathematical object, it has many applications for other sciences such as physics, biology, economics, geophysics, medicine, and bioengineering [8, 10-13, 15, 21]. Different approaches to fractional calculus use different definitions of the fractional derivative, some of the most notable of which are provided by Riemann-Liouville, Caputo, Ries, Feller, Caputo-Fabrizio, Atangana, Baleanu etc.

In particular, there has been great interest in the fractional Riemann-Liouville operator because of its simplicity and the importance of obtaining many useful results when combining it with other operators. However, the very complicated mathematical manipulation in addition with the fact that most of the fractional approaches deal mainly with the presence of non-local fractional differential operators makes it poorly understood by much of the scientific community. Recently, researchers have become increasingly interested in fractional calculus in the context of bicomplex analysis [14, 18].

In the present work, we consider the Riemann-Liouville integral and differential operators of bicomplex order and find the bicomplex Laplace transform for each operator. The aim of the work is to present the modification in analysis of Laplace transform applied on generalized fractional operators that admits several useful results in framework of bicomplex-fractional analysis. For this purpose, to apply the bicomplex Laplace transform, we use several of the results discussed in [18, 24]. A detailed discussion of fractional calculus can be found in [20, 22, 23, 31, 33, 35] and for the bicomplex analysis one can go through [30, 32].

The development of the paper is as follows: In Section 2, we give a brief discussion of bicomplex number system, fractional calculus, bicomplex Laplace transform, and Riemann-Liouville integration and differentiation of bicomplex order of bicomplex-valued functions of real variable. Section 3 contains the main work in which we find bicomplex Laplace transform of Riemann-Liouville fractional operators of bicomplex order and obtain various useful results. In Section 4, we give an application of our work by finding voltage V and current I in an LCR-circuit. Section 5 concludes our work.

2. Preliminaries

In this section, we discuss some fundamental theory of the bicomplex number system and fractional calculus. For this purpose, we adopt most of the notations and concepts from [29] and [30] for bicomplex space and fractional calculus, respectively throughout the paper.

2.1 Basic facts of bicomplex numbers

A bicomplex number can be seen as ordered pair of two complex numbers, moreover a 4-dimensional real vector space

$$\mathbb{C}_2 = \{w\colon w = z_1 + jz_2 = x_1 + iy_1 + jx_2 + ijy_2\},$$

where $z_m = x_m + iy_m\colon x_m, y_m \in \mathbb{R}\,; m = 1,2$ and further z_1 and z_2 can be treated as bi-real and bi-imaginary part of w, respectively. The imaginary units i, j, and k obeys the following

$$i^2 = j^2 = -1,\;\; ij = ji = k,\; k^2 = 1.$$

Corresponding to these three imaginary units i, j, and k we can identify the below subsets of $\mathbb{C}_2$

$$\mathbb{C}(i) = \{z : z = x_1 + iy_1 \mid x_1, y_1 \in \mathbb{R}\},$$
$$\mathbb{C}(j) = \{\zeta : \zeta = x_1 + jx_2 \mid x_1, x_2 \in \mathbb{R}\},$$
$$\mathbb{D} = \{\rho : \rho = x_1 + ky_2 \mid x_1, y_2 \in \mathbb{R}\}.$$

Both $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields of complex numbers and $\mathbb{D}$ is set of hyperbolic numbers. We can visualize $\mathbb{C}_2$ as complexification of $\mathbb{C}(j)$. The *idempotent representation*, which is unique for every bicomplex number $w = z_1 + jz_2$, can be seen as

$$w = (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2 = w_1 e_1 + w_2 e_2, \tag{1}$$

where $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$ which satisfy the identities $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$, $e_1 - e_2 = ij$, and $e_1 e_2 = e_2 e_1 = 0$. The last two identities give an important fact that e_1 and e_2 are zero divisors in $\mathbb{C}_2$ and hence, $\mathbb{C}_2$ is neither a division algebra nor a field. The set of all zero divisor in $\mathbb{C}_2$ can be given as

$$\mathcal{O}_2 = \{\lambda(1 \pm ij) : \lambda \in \mathbb{C}(i) \setminus \{0\}\}$$

known as null cone. Three conjugations w.r.t. i, j, and k [29, p. 8] are defined as

$$\bar{w} = \bar{z}_1 + j\bar{z}_2 = x_1 - iy_1 + jx_2 - ijy_2 = \bar{w}_2 e_1 + \bar{w}_1 e_2,$$
$$w^\dagger = z_1 - jz_2 = x_1 + iy_1 - jx_2 - ijy_2 = w_2 e_1 + w_1 e_2,$$
$$w^* = \bar{w}^\dagger = \bar{w}^\dagger = \bar{z}_1 - j\bar{z}_2 = x_1 - iy_1 - jx_2 + ijy_2 = \bar{w}_1 e_1 + \bar{w}_2 e_2,$$

respectively. Considering the real representation, the real and three imaginary parts of a bicomplex number $w = x_1 + iy_1 + jx_2 + ijy_2$ are calculated by

$$x_1 = \frac{1}{4}\left(w + \bar{w} + w^\dagger + w^*\right)$$
$$y_1 = \frac{1}{4i}\left(w - \bar{w} + w^\dagger - w^*\right)$$
$$x_2 = \frac{1}{4j}\left(w + \bar{w} - w^\dagger - w^*\right)$$
$$y_2 = \frac{1}{4k}\left(w - \bar{w} - w^\dagger + w^*\right).$$

Using the projection mappings $\mathcal{P}_1$, $\mathcal{P}_2 : \mathbb{C}_2 \to \mathbb{C}(i)$, which are projections of $\mathbb{C}_2$ onto $\mathbb{C}(i)$; another interpretation for w can be considered as

$$w = \mathcal{P}_1(w)e_1 + \mathcal{P}_2(w)e_2,$$

where

$$\mathcal{P}_1(z_1 + jz_2) = z_1 - iz_2 \tag{2}$$
$$\mathcal{P}_2(z_1 + jz_2) = z_1 + iz_2. \tag{3}$$

Through the idempotent representation (1), we can give some elementary operations as follows:

$$\begin{aligned} w+\xi &= (w_1+\xi_1)e_1+(w_2+\xi_2)e_2 \\ w\xi &= w_1\xi_1e_1+w_2\xi_2e_2 \\ w^n &= w_1^ne_1+w_2^ne_2 \\ e^w &= e^{w_1}e_1+e^{w_2}e_2. \end{aligned}$$

Let X be a domain in $\mathbb{C}_2$, then using projections (2) and (3) the corresponding domains in the complex plane are given as

$$\begin{aligned} X_1 &= \mathcal{P}_1(X) = \{w_1 = z_1 - iz_2 : z_1, z_2 \in \mathbb{C}(i)\} \\ X_2 &= \mathcal{P}_2(X) = \{w_2 = z_1 + iz_2 : z_1, z_2 \in \mathbb{C}(i)\}, \end{aligned}$$

which determines, X in $\mathbb{C}_2$ as

$$X = \{(w_1, w_2) : w_1 \in X_1, w_2 \in X_2\} = X_1 \times X_2.$$

The Euclidean norm in $\mathbb{R}^4$ for $w = z_1 + jz_2 \in \mathbb{C}_2$, is defined as

$$\| w \|_2 = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2},$$

where ' $|\cdot|$ ' denotes Euclidean norm in complex space. It can be easily proved that

$$\| \xi w \|_2 \leq \sqrt{2} \, \| \xi \|_2 \| w \|_2.$$

A sequence in $\mathbb{C}_2$ (a bicomplex sequence) is a function defined by $w_n : \mathbb{N} \to \mathbb{C}_2$, $n \mapsto w_n$. This sequence converges to a point $w_0 \in \mathbb{C}_2$ if and only if for every $\varepsilon > 0$ there is a $k(\varepsilon) \in \mathbb{N}$ such that $\|w_n - w_0\|_2 < \varepsilon$, $\forall n \geq k(\varepsilon)$. The sequence w_n is a Cauchy sequence in $\mathbb{C}_2$ (a bicomplex Cauchy sequence) if and only if for every $\varepsilon > 0$ there is a $k(\varepsilon) \in \mathbb{N}$ such that $\|w_n - w_m\|_2 < \varepsilon$, $\forall n, m \geq k(\varepsilon)$. Also, w_n converges to a point in $\mathbb{C}_2$ if and only if it is a bicomplex Cauchy sequence. For $\xi = z_3 + jz_4 \approx (z_3, z_4)$, we consider bicomplex function

$$f(\xi) = f(z_3, z_4) = u(z_3, z_4) + jv(z_3, z_4) = \big(u(z_3, z_4), v(z_3, z_4)\big)$$

and let Υ be a four dimensional piecewise continuously differentiable curve in a set $S \subseteq \mathbb{C}_2$. Then the bicomplex integration of bicomplex function f is defined as a line integral, that is evaluated with respect to some four-dimensional curve Υ in $\mathbb{C}_2$. More specifically, the bicomplex integration is defined as

$$\int_\Upsilon f(\xi)d\xi, \quad d\xi = (dz_3, dz_4). \tag{4}$$

If we represent Υ in parametric form $\xi(t) = (z_3(t), z_4(t))$, where $r \leq t \leq s$. Then (4) can be rewritten as

$$\int_\Upsilon f(\xi)d\xi = \int_r^s f\big(\xi(t)\big)\xi'(t)dt.$$

Here $\xi'(t)$ may discontinuous at some points. Υ can be taken as a curve made up of two component curves γ_1 and γ_2 in $\mathbb{C}(i)$ i.e.

$\Upsilon = (\gamma_1, \gamma_2)$.

On curve Υ, (4) can be determined as

$$\int_{\Upsilon} f(\xi)d\xi = \left\{\int_{\gamma_1} f_1(\xi_1)d\xi_1\right\}e_1 + \left\{\int_{\gamma_2} f_2(\xi_2)d\xi_2\right\}e_2$$

A very basic and foundational background for knowing and understanding the bicomplex space and its geometric interpretation can be found in [29, 32].

2.2 Bicomplex Riemann-Liouville fractional operators

From the theory of fractional calculus, we need bicomplex Riemann-Liouville fractional operators to continue further discussion. We begin our discussion by giving a brief description of these fractional operators. In [18], bicomplex Riemann-Liouville fractional derivative and integral operator are defined as follows:

Definition 2.1 (Riemann-Liouville integral of bicomplex order). Let $w = z_1 + jz_2 \in \mathbb{C}_2$ with $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$ and f be piecewise continuous on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$. Then for $t > 0$

$$_0D_t^{-w} f(t) = \frac{1}{\Gamma_2(w)} \int_0^t f(x)(t-x)^{w-1}dx, \tag{5}$$

where Γ_2 is bicomplex gamma function [19]. Let us denote $\mathcal{C}$ as the class of functions defined in above Definition 2.1 which will be called bicomplex locally integrable functions.

Definition 2.2 (Riemann-Liouville derivative of bicomplex order). Let f be a function of class $\mathcal{C}$ and let $w \in \mathbb{C}_2$ with $\mathrm{Re}(z_1) > 0$. Let $m = \lfloor \mathrm{Re}(z_1) \rfloor + 1$. Then the Riemann-Liouville fractional derivative of f of order w is

$$_0D_t^{w} f(t) = {}_0D_t{}^{m}\, {}_0D_t^{-(m-w)} f(t) = \frac{1}{\Gamma_2(m-w)} \frac{d^m}{dt^m} \int_0^t f(x)(t-x)^{m-w-1}dx. \tag{6}$$

This work attained some of important results as

$$\begin{aligned} D^{w} &\equiv e_1 D^{w_1} + e_2 D^{w_2} \\ D^{-w} &\equiv e_1 D^{-w_1} + e_2 D^{-w_2}. \end{aligned} \tag{7}$$

2.3 Bicomplex Laplace transform

Recall that a bicomplex-valued function of variable t defined in $J' = (0, \infty)$ is said to be of exponential order K if there exist $M > 0$ and $T > 0$ such that $e^{-Kt}|f(t)|_2 \le M$, for all $t \ge T$. Now, if $f(t)$ is of exponential order K, then its bicomplex Laplace transform [24] is given by

$$\mathcal{L}[f(t); \xi] = \mathcal{F}(\xi) = \int_0^{\infty} f(x)e^{-\xi x}dx,$$

where $\xi = z_3 + jz_4$ with $\mathrm{Re}(z_3) > K + |\mathrm{Im}(z_4)|$. Here, $\mathcal{F}(\xi)$ exists and convergent for all ξ which has hyperbolic projection of ξ i.e. $H_p(\xi)$ in the right half plane $\mathrm{Re}(z_3) > K +$

$|\text{Im}(z_4)|$. There are infinite ξ which have the same hyperbolic projection because $\text{Im}(z_3)$ and $\text{Re}(z_4)$ are free from restrictions.

One of the most useful properties of the Laplace transform is embedded in the convolution theorem. The theorem states that the Laplace transform of the convolution of two functions is the product of their Laplace transform. Thus, if $\mathcal{F}(\xi)$ and $\mathcal{G}(\xi)$ are the bicomplex Laplace transforms of $f(t)$ and $g(t)$, respectively, then

$$\mathcal{L}\left\{\int_0^t f(t-x)g(x)dx\right\} = \mathcal{F}(\xi)\mathcal{G}(\xi).$$

Considering idempotent representation, the following results can be proved for bicomplex Laplace transform:

$$\mathcal{L}[f(t);\ \xi] = L[f_1(t);\ \xi_1]e_1 + L[f_2(t);\ \xi_2]e_2 \tag{8}$$

$$\Rightarrow \mathcal{F}(\xi) = \{\mathcal{F}_1(\xi_1)\}e_1 + \{\mathcal{F}_2(\xi_2)\}e_2 \tag{9}$$

$$\begin{aligned}\mathcal{L}[f(t);\ \xi] &= \mathcal{L}[f_1(t)e_1 + f_2(t)e_2;\ \xi_1 e_1 + \xi_2 e_2]\\ &= L[f_1(t);\ \xi_1]e_1 + L[f_2(t);\ \xi_2]e_2,\end{aligned} \tag{10}$$

where $L[f_r(t);\ \xi_r]; r = 1,2$ are complex Laplace transforms w.r.t variable ξ_r, respectively.

2.4 The Laplace transform of Riemann-Liouville operators in complex space

Before going in the analysis of bicomplex Laplace transform of bicomplex Riemann Liouville operator, let us first review some well-known established results of the Laplace transform of Riemann-Liouville operators in complex space [30].

Let $f(t)$ be a function of exponential order k defined on interval $(0,\infty)$ and $F(s)$ denote the Laplace transform of it i.e. $F(s) = L\{f(t);\ s\}$, where L denoted Laplace transform in complex space. Let $z = x + iy \in \mathbb{C}$ with $\text{Re}(z) > 0$. Then the Laplace transform of the Riemann-Liouville integral of order z [30, p.69] is defined as

$$L[D^{-z}f(t);s] = s^{-z}F(s),\ \text{Re}(z) > 0.$$

The Laplace transform of the Riemann-Liouville integral of the derivative of $f(t)$ is given by

$$L[D^{-z}(Df(t));\ s] = s^{-z}[sF(s) - f(0)],\ \text{Re}(z) > 0.$$

The Laplace transform of the derivative of the Riemann-Liouville integral is then defined as

$$L[D(D^{-z}f(t));\ s] = s^{1-z}F(s),\ \ \text{Re}(z) > 0.$$

In a similar way, with the considerations that $f(t)$ is of the form

$$t^\nu \eta(t) \text{ or } t^\nu \log(t)\,\eta(t),$$

where $\nu > -1$ and $\eta(t) = \sum_{n=0}^{\infty} a_n t^n$ has a radius of convergence $R > 0$. The Laplace transform of the Riemann-Liouville derivative is given as

$$L[D^z f(t);\ s] = s^z F(s),\ \text{Re}(z) < \nu + 1, \qquad f(t) = t^\nu \eta(t).$$

In the next section, we will find the bicomplex Laplace transform of bicomplex Riemann-Liouville operators as well as some other important results in the bicomplex space.

3. Bicomplex Laplace Transform of the Riemann-Liouville Operators of Bicomplex Order

The bicomplex Laplace transform will prove to be an essential tool in our study of the bicomplex fractional differential equation. We briefly open up our discussion of this powerful method in the current section. The fractional integral and differentiation of bicomplex order defined in (5) and (6) are convolution integrals. This fact prompts us to discuss the bicomplex Laplace transform of these bicomplex fractional operators.

3.1 Bicomplex Laplace transform of the Riemann-Liouville integral operator of bicomplex order

Theorem 3.1. Let f be a function of class $\mathcal{C}$ such that $D^{-w}f(t)$ is of exponential order K, then for $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$ and $\mathrm{Re}(z_3) > K + |\mathrm{Im}(z_3)|$, we have

$$\mathcal{L}[D^{-w}f(t);\ \xi] = \xi^{-w}\mathcal{F}(\xi), \tag{11}$$

Proof. Taking (7) and (8)-(10) into account, we have

$$\begin{aligned}
\mathcal{L}[D^{-w}f(t);\ \xi] &= \mathcal{L}[(e_1D^{-w_1} + e_2D^{-w_2})(f_1(t)e_1 + f_2(t)e_2);\ \xi] \\
&= \mathcal{L}[D^{-w_1}f_1(t)e_1 + D^{-w_2}f_2(t)e_2;\ \xi_1e_1 + \xi_2e_2] \\
&= L[D^{-w_1}f_1(t); \xi_1]e_1 + L[D^{-w_2}f_2(t);\ \xi_2]e_2 \\
&= L\left[\frac{1}{\Gamma(w_1)}\int_0^t (t-x)^{w_1-1}f_1(x)dx;\ \xi_1\right]e_1 \\
&\quad + L\left[\frac{1}{\Gamma(w_2)}\int_0^t (t-x)^{w_2-1}f_2(x)dx;\ \xi_2\right]e_2 \\
&= \left\{\frac{1}{\Gamma(w_1)}L[t^{w_1-1};\ \xi_1]L[f_1(t);\ \xi_1]\right\}e_1 \\
&\quad + \left\{\frac{1}{\Gamma(w_2)}L[t^{w_2-1};\ \xi_2]L[f_2(t);\ \xi_2]\right\}e_2 \\
&= \left[\xi_1^{-w_1}\mathcal{F}_1(\xi_1)\right]e_1 + \left[\xi_2^{-w_2}\mathcal{F}_2(\xi_2)\right]e_2 \\
&= \left\{\xi_1^{-w_1}e_1 + \xi_2^{-w_2}e_2\right\}\{\mathcal{F}_1(\xi_1)e_1 + \mathcal{F}_2(\xi_2)e_2\} \\
&= \xi^{-w}\mathcal{F}(\xi).
\end{aligned} \tag{12}$$

Remark 3.1. (11) is valid even if $w = 0$, but (12) is indeterminate. We now try to find the Laplace transform of the Riemann-Liouville integration of some elementary bicomplex-valued functions of real variable calculated in Table 2 of [18]. For instance,

$$D^{-w}t^u = \frac{\Gamma_2(u+1)}{\Gamma_2(u+w+1)}t^{u+w},$$

which provides

$$
\begin{aligned}
\mathcal{L}[D^{-w}t^{u};\ \xi] &= \mathcal{L}\left[\frac{\Gamma_2(u+1)}{\Gamma_2(u+w+1)}t^{u+w};\ \xi\right] \\
&= L\left[\frac{\Gamma(u+1)}{\Gamma(u+w_1+1)}t^{u+w_1};\ \xi_1\right]e_1 + L\left[\frac{\Gamma(u+1)}{\Gamma(u+w_2+1)}t^{u+w_2};\ \xi_2\right]e_2 \\
&= \left[\frac{\Gamma(u+1)}{\xi_1^{u+w_1+1}}\right]e_1 + \left[\frac{\Gamma(u+1)}{\xi_2^{u+w_2+1}}\right]e_2 \\
&= \frac{\Gamma_2(u+1)}{\xi^{u+w+1}},
\end{aligned}
$$

where $u = \varrho_1 + j\varrho_2 = u_1e_1 + u_2e_2 \in \mathbb{C}_2$ such that $\mathrm{Re}(\varrho_1) > |\mathrm{Im}(\varrho_2)| - 1$, $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$, and $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$. In the similar manner, for $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$, we can derive the following:

(i) $\boldsymbol{f(t) = e^{at}}$

$l\mathcal{L}[D^{-w}e^{at};\ \xi] = \mathcal{L}[E_t(w,a);\xi]$

$$= \frac{1}{\xi^{w}(\xi - a)};\ a \in \mathbb{C}_2, \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|,$$

where $E_t(w,a) = t^w \sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma_2(w+k+1)}$.

(ii) $\boldsymbol{f(t) = \sin at}$

$\mathcal{L}[D^{-w}\sin at\,;\xi] = \mathcal{L}[S_t(w,a);\ \xi]$

$$= \frac{1}{\xi^{w}(\xi^2 + a^2)};\ a \in \mathbb{C}_2, \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|,$$

where $S_t(w,a) = t^w \sum_{k_{\text{odd}}}^{\infty} \frac{(-1)^{(k-1)/2}(at)^k}{\Gamma_2(w+k+1)}$.

(iii) $\boldsymbol{f(t) = \cos at}$

$\mathcal{L}[D^{-w}\cos at\,;\xi] = \mathcal{L}[C_t(w,a);\ \xi]$

$$= \frac{1}{\xi^{w-1}(\xi^2 + a^2)};\ a \in \mathbb{C}_2, \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|,$$

where $C_t(w,a) = t^w \sum_{k_{\text{even}}}^{\infty} \frac{(-1)^{k/2}(at)^k}{\Gamma_2(w+k+1)}$.

(iv) $\boldsymbol{f(t) = t^{u-1}e^{at}}$

$$\mathcal{L}[D^{-w}t^{u-1}e^{at};\xi] = \frac{\Gamma_2(u)}{\xi^{w}(\xi - a)^{u}};\ a \in \mathbb{C}_2, \mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|,\ \mathrm{Re}(\varrho_1) > |\mathrm{Im}(\varrho_2)|.$$

Now, we discuss the bicomplex Laplace transform of the bicomplex Riemann-Liouville fractional integral in a different way. We begin our discussion by exploring the bicomplex Laplace transform of the bicomplex Riemann-Liouville integral of the derivative and the bicomplex Laplace transform of the derivative of the bicomplex Riemann-Liouville fractional integral. Suppose that f is continuous on J and Df is of class $\mathcal{C}$ and of exponential order K then,

$$\begin{aligned}\mathcal{L}[D^{-w}(Df(t));\xi] &= \xi^{-w}\mathcal{L}[Df(t);\xi] \\ &= \xi^{-w}[\xi\mathcal{F}(\xi)-f(0))],\ \mathrm{Re}(z_1)>|\mathrm{Im}(z_2)|.\end{aligned}$$

Since we supposed $f(t)$ to be continuous on J, hence $f(0)$ exists. Thus, we have found the bicomplex Laplace transform of the Riemann-Liouville integral of bicomplex order of the derivative. This formula is obviously valid if $w=0$.

Further, we consider the problem of finding the bicomplex Laplace transform of the derivative of the Riemann-Liouville integral of bicomplex order.

$$\begin{aligned}\mathcal{L}[D(D^{-w}f(t));\ \xi] &= \mathcal{L}[D^{-w}(Df(t)));\xi]+f(0)\mathcal{L}\left[\frac{t^{w-1}}{\Gamma_2(w)};\ \xi\right] \\ &= \xi^{-w}[\xi\mathcal{F}(\xi)-f(0)]+\xi^{-w}f(0) \\ &= \xi^{1-w}\mathcal{F}(\xi);\ \mathrm{Re}(z_1)>|\mathrm{Im}(z_2)|. \end{aligned} \tag{13}$$

Now, if $w=0$,

$$\mathcal{L}[Df(t);\ \xi]=\xi\mathcal{F}(\xi)-f(0).$$

But this is not the same result, we would get if we let $w=0$ in (13). This arises from the following facts:

$$\mathcal{L}\left\{\lim_{w\to 0}\frac{t^{w-1}}{\Gamma_2(w)}\right\}=0,$$

and

$$\lim_{w\to 0}\mathcal{L}\left\{\frac{t^{w-1}}{\Gamma_2(w)}\right\}=1$$

i.e. ' $\mathcal{L}$ ' and 'lim' do not commute. With little effort, we see that with the aid of bicomplex analysis to the the fractional calculus, we have found the bicomplex Laplace transforms of some non-elementary bicomplex-valued functions of real variable.

3.2 Bicomplex Laplace Transform of the Riemann-Liouville differential operator of bicomplex order

In the previous subsection we introduced the bicomplex Laplace transform and found the bicomplex Laplace transform of the Riemann-Liouville fractional integral of bicomplex order. We continue our discussion by investigating the bicomplex Laplace transform of the Riemann-Liouville fractional derivatives of bicomplex order. To fulfill this purpose, let us define a new class of functions $\mathcal{C}$ which contains all the functions $f(t)$ of the form $t^{\lambda}\eta(t)$ or $t^{\lambda}\log(t)\,\eta(t)$,

where $\lambda > -1,\ \eta(t) = \sum_{n=0}^{\infty} a_n t^n$ has a radius of convergence $R > 0$, and $a_k \in \mathbb{C}_2,\ k \in \mathbb{Z}^+ \cup \{0\}$.

Let $w = z_1 + jz_2$ with $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_1) < |\mathrm{Im}(z_2)| + \lambda + 1$. If $f(t) = t^{\lambda}\eta(t)$, then we have

$$D^w f(t) = t^{\lambda-w} \sum_{n=0}^{\infty} a_n \frac{\Gamma_2(n+\lambda+1)}{\Gamma_2(n+\lambda+1-w)} t^n, \tag{14}$$

and if $f(t) = t^{\lambda}\log(t)\eta(t)$, then

$$\begin{aligned} D^w f(t) = \;& t^{\lambda-w}(\log t) \sum_{n=0}^{\infty} a_n \frac{\Gamma_2(n+\lambda+1)}{\Gamma_2(n+\lambda+1-w)} t^n \\ & + t^{\lambda-w} \sum_{n=0}^{\infty} a_n [\Psi(n+\lambda+1) - \Psi(n+\lambda+1-w)] \times \frac{\Gamma_2(n+\lambda+1)}{\Gamma_2(n+\lambda+1-w)} t^n, \end{aligned} \tag{15}$$

where Ψ is bicomplex digamma functions and the hypothesis $\mathrm{Re}(z_1) < |\mathrm{Im}(z_2)| + \lambda + 1$ shows that (14) and (15) exist. Moreover, $D^w f(t) \in \mathcal{C}$ in both cases. If $D^w f(t)$ also is of exponential order, its bicomplex Laplace transform exists.

Lemma 3.2. If f is given by $f(t) = t^{\lambda}\eta(t)$, then for $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$, we have

$$\mathcal{F}(\xi) = \frac{1}{\xi^{\lambda+1}} \sum_{n=0}^{\infty} a_n \Gamma(n+\lambda+1) \xi^{-n}. \tag{16}$$

Proof. We have

$$\begin{aligned} \mathcal{F}(\xi) &= \mathcal{L}[f(t)] \\ &= \mathcal{L}[t^{\lambda}\eta(t); \xi] \\ &= \mathcal{L}\left[t^{\lambda} \sum_{n=0}^{\infty} a_n t^n; \xi\right] \\ &= \mathcal{L}\left[\sum_{n=0}^{\infty} a_n t^{\lambda+n}; \xi\right] \\ &= L\left[\sum_{n=0}^{\infty} {}_1a_n t^{\lambda+n}; \xi_1\right] e_1 + L\left[\sum_{n=0}^{2} a_n t^{\lambda+n}; \xi_2\right] e_2 \\ &= \left[\frac{1}{\xi_1^{\lambda+1}} \sum_{n=0}^{\infty} {}_1a_n \Gamma(n+\lambda+1) \xi_1^{-n}\right] e_1 + \left[\frac{1}{\xi_2^{\lambda+1}} \sum_{n=0}^{\infty} 2_2 a_n \Gamma(n+\lambda+1) \xi_2^{-n}\right] e_2 \\ &= \frac{1}{\xi^{\lambda+1}} \sum_{n=0}^{\infty} a_n \Gamma(n+\lambda+1) \xi^{-n}. \end{aligned}$$

In this case, bicomplex Laplace transform of the Riemann-Liouville derivative of bicomplex order can be determined as

Theorem 3.3. If f is given by $f(t) = t^{\lambda}\eta(t)$, then for $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$

$$\mathcal{L}[D^w f(t);\xi] = \xi^w \mathcal{F}(\xi). \tag{17}$$

Proof. With the help of (14) and (16), we can calculate

$$\begin{aligned}
\mathcal{L}[D^w f(t);\xi] &= \mathcal{L}\left[t^{\lambda-w}\sum_{n=0}^{\infty} a_n \frac{\Gamma_2(n+\lambda+1)}{\Gamma_2(n+\lambda+1-w)} t^n;\ \xi\right] \\
&= L\left[\sum_{n=0}^{\infty} {}_1a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-w_1)} t^{n+\lambda-w_1};\ \xi_1\right] e_1 \\
&\quad + L\left[\sum_{n=0}^{\infty} {}_2a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-w_2)} t^{n+\lambda-w_2};\ \xi_2\right] e_2 \\
&= \left[\sum_{n=0}^{\infty} {}_1a_n \frac{\Gamma(n+\lambda+1)}{\xi_1^{n+\lambda-w_1+1}}\right] e_1 + \left[\sum_{n=0}^{\infty} {}_2a_n \frac{\Gamma(n+\lambda+1)}{\xi_2^{n+\lambda-w_2+1}}\right] e_2 \\
&= \sum_{n=0}^{\infty} a_n \frac{\Gamma_2(n+\lambda+1)}{\xi^{n+\lambda-w+1}} \\
&= \xi^w \mathcal{F}(\xi).
\end{aligned}$$

In the similar manner, we can state

Lemma 3.4. If $f(t) = t^{\lambda}\log(t)\eta(t)$ then for $t > 0$ and $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$, we have

$$\mathcal{F}(\xi) = \frac{1}{\xi^{\lambda+1}}\sum_{n=0}^{\infty} a_n \Gamma_2(n+\lambda+1)\{\Psi(n+\lambda+1) - \log(\xi)\}\xi^{-n}. \tag{18}$$

Proof. The proof can be easily achieved as

$$\begin{aligned}
\mathcal{F}(\xi) &= \mathcal{L}[f(t);\xi] \\
&= \mathcal{L}\left[\sum_{n=0}^{\infty} \log(t) a_n t^{\lambda+n};\xi\right] \\
&= L\left[\sum_{n=0}^{\infty} \log(t)\, {}_1a_n t^{\lambda+n};\xi_1\right] e_1 + L\left[\sum_{n=0}^{\infty} \log(t)\, {}_2a_n t^{\lambda+n};\xi_2\right] e_2 \\
&= \left[\frac{1}{\xi_1^{\lambda+1}}\sum_{n=0}^{\infty} {}_1a_n \Gamma(n+\lambda+1)\{\psi(n+\lambda+1) - \log(\xi_1)\}\xi_1^{-n}\right] e_1 \\
&\quad + \left[\frac{1}{\xi_2^{\lambda+1}}\sum_{n=0}^{\infty} {}_2a_n \Gamma(n+\lambda+1)\{\psi(n+\lambda+1) - \log(\xi_2)\}\xi_2^{-n}\right] e_2
\end{aligned}$$

$$= \frac{1}{\xi^{\lambda+1}} \sum_{n=0}^{\infty} a_n \Gamma_2(n+\lambda+1)\{\Psi(n+\lambda+1) - \log(\xi)\}\xi^{-n},$$

where ψ is complex digamma function.

With the result (18) the bicomplex Laplace transform of the Riemann-Liouville derivative of bicomplex order can be defined as

Theorem 3.5. If $f(t) = t^{\lambda}\log(t)\eta(t)$ then for $t > 0$ and $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$, we have

$$\mathcal{L}[D^w f(t); \xi] = \xi^w \mathcal{F}(\xi). \tag{19}$$

Proof. Using (15) and (18), we have

$$\mathcal{L}[D^w f(t);\ \xi]$$

$$= \mathcal{L}\Bigg[t^{\lambda-w}(\log t) \sum_{n=0}^{\infty} a_n \frac{\Gamma_2(n+\lambda+1)}{\Gamma_2(n+\lambda+1-w)} t^n$$

$$+ t^{\lambda-w} \sum_{n=0}^{\infty} a_n [\Psi(n+\lambda+1) - \Psi(n+\lambda+1-w)]$$

$$\times \frac{\Gamma_2(n+\lambda+1)}{\Gamma_2(n+\lambda+1-w)} t^n;\ \xi\Bigg]$$

$$= L\Bigg[t^{\lambda-w_1}(\log t) \sum_{n=0}^{\infty} {}_1a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-w_1)} t^n$$

$$+t^{\lambda-w_1} \sum_{n=0}^{\infty} {}_1a_n [\psi(n+\lambda$$

$$- \psi(n+\lambda+1-w_1)] \times \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-w_1)} t^n;\ \xi_1\Bigg] e_1 +$$

$$L\Bigg[t^{\lambda-w_2}(\text{l} \ \ \text{og}\, t) \sum_{n=0}^{\infty} {}_2a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-w_2)} t^n$$

$$+t^{\lambda-w_1} \sum_{n=0}^{\infty} {}_1a_n [\psi(n+\lambda+1)$$

$$- \psi(n+\lambda+1-w_1)] \times \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-w_2)} t^n;\ \xi_2\Bigg] e_2$$

$$= \left[-\frac{\log \xi_1}{\xi_1^{\lambda-w_1+1}} \sum_{n=0}^{\infty} {}_1a_n \Gamma(n+\lambda+1)\xi_1^{-n} + \frac{1}{\xi_1^{\lambda-w_1+1}} \sum_{n=0}^{\infty} {}_1a_n \Gamma(n+\lambda+1)\psi(n+\lambda+1)\xi_1^{-n}\right] e_1$$

$$+ \left[-\frac{\log \xi_2}{\xi_2^{\lambda-w_2+1}} \sum_{n=0}^{\infty} {}_2a_n \Gamma(n+\lambda+1)\xi_2^{-n} + \frac{1}{\xi_2^{\lambda-w_2+1}} \sum_{n=0}^{\infty} {}_2a_n \Gamma(n+\lambda+1)\psi(n+\lambda+1)\xi_2^{-n}\right] e_2$$

$$= - \frac{\log \xi}{\xi^{\lambda-w+1}} \sum_{n=0}^{\infty} a_n \Gamma_2(n+\lambda+1)\xi^{-n} + \frac{1}{\xi^{\lambda-w+1}} \sum_{n=0}^{\infty} a_n \Gamma_2(n+\lambda+1)\Psi(n+\lambda+1)\xi^{-n}$$

$$= \xi^w \mathcal{F}(\xi).$$

Remark 3.2. If $\mathrm{Re}(z_1) < 0$, (17) and (19) are just the statement that $\xi^w \mathcal{F}(\xi)$ is the bicomplex Laplace transform of the Riemann-Liouville integral of bicomplex order-a result we established in (11). Furthermore, (19) certainly is true if $w = 0$, a case also covered by (19) since $\lambda > -1$.

One should observe that the Riemann-Liouville integral of bicomplex order of a function of class $\mathcal{C}$ is again of class $\mathcal{C}$ but the Riemann-Liouville derivative of bicomplex order of a function of class $\mathcal{C}$ need not be of class $\mathcal{C}$. Thus, if we desire $D^w f$ to be of class $\mathcal{C}$, we must require that $\mathrm{Re}(z_1) < |\mathrm{Im}(z_2)| + \lambda + 1$.

Let $w = z_1 + jz_2 = w_1 e_1 + w_2 e_2 \in \mathbb{C}_2$ with $\mathrm{Re}(z_1) > 0, m_1 = \lfloor \mathrm{Re}(w_1) \rfloor + 1$, $m_2 = \lfloor \mathrm{Re}(w_2) \rfloor + 1$, and $m = \max\{m_1, m_2\}$. Now, let us assume for the moment that the bicomplex Laplace transform of $f(t)$ exists. Then for $\mathrm{Re}(z_3) > K + |\mathrm{Im}(z_4)|$, we have

$$\begin{aligned} \mathcal{L}[D^w f(t); \xi] &= \mathcal{L}[(e_1 D^{w_1} + e_2 D^{w_2})(f_1(t)e_1 + f_2(t)e_2);\ \xi] \\ &= \mathcal{L}[D^{w_1} f_1(t)e_1 + D^{w_2} f_2(t)e_2; \xi] \\ &= L[D^{w_1} f_1(t);\ \xi_1]e_1 + L[D^{w_2} f_2(t);\ \xi_2] \\ &= L\left[D^m\{D^{-(m-w_1)}\} f_1(t);\ \xi_1\right]e_1 + L\left[D^m\{D^{-(m-w_2)} f_2(t)\};\ \xi_2\right]e_2. \quad (20) \end{aligned}$$

Now,

$$\begin{aligned} L\left[D^m\{D^{-(m-w_1)}\} f_1(t);\ \xi_1\right] &= \xi_1^m L\left[D^{-(m-w_1)} f_1(t);\ \xi_1\right] - \sum_{n=0}^{m-1} \xi_1^{m-n-1} D^n\{D^{-(m-w_1)} f_1(t)\}\Big|_{t=0} \\ &= \xi_1^m \left\{\xi_1^{-(m-w_1)} \mathcal{F}_1(\xi_1)\right\} - \sum_{n=0}^{m-1} \xi_1^{m-n-1} D^{n-(m-w_1)} f_1(0) \end{aligned}$$

$$= \xi_1^{w_1} \mathcal{F}_1(\xi_1) - \sum_{n=0}^{m-1} \xi_1^{m-n-1} D^{n-m+w_1} f_1(0). \quad (21)$$

Similarly, we can find

$$L\left[D^m\{D^{-(m-w_2)}\}f_2(t);\xi\right]$$
$$= \xi_2^{w_2}\mathcal{F}_2(\xi_2) - \sum_{n=0}^{m-1} \xi_2^{m-n-1}D^{n-m+w_2}f_1(0) \quad (22)$$

On substituting values from (21) and (22) into (20), we have

$$\mathcal{L}[D^w f(t);\xi] = \left[\xi_1^{w_1}\mathcal{F}_1(\xi_1) - \sum_{n=0}^{m-1} \xi_1^{m-n-1}D^{n-m+w_1}f_1(0)\right]e_1 + \left[\xi_2^{w_2}\mathcal{F}_2(\xi_2) - \sum_{n=0}^{m-1} \xi_2^{m-n-1}D^{n-m+w_2}f_2(0)\right]e_2$$

$$= (\xi_1e_1+\xi_2e_2)^{w_1e_1+w_2e_2}(\mathcal{F}_1(\xi_1)e_1+\mathcal{F}_2(\xi_2)e_2) - \sum_{n=0}^{m-1}(\xi_1e_1+\xi_2e_2)^{(m-n-1)}\left(e_1D^{(n-m+w_1)}+e_2D^{(n-m+w_2)}\right)(f_1(0)e_1+f_2(0)e_2)$$
$$= \xi^w\mathcal{F}(\xi) - \sum_{n=0}^{m-1} \xi^{m-n-1}D^{n-m+w}f(0). \quad (23)$$

Thus, we have found the bicomplex Laplace transform of the Riemann-Liouville fractional derivative of bicomplex order. We have calculated Riemann-Liouville fractional differentiation and integration of bicomplex order in Table 2 of [18]. From that table, we have

$$D^w t^u = \frac{\Gamma_2(u+1)}{\Gamma_2(u-w+1)}t^{u-w},$$

which provide us the bicomplex Laplace transform as follows:

$$\begin{aligned}\mathcal{L}[D^w t^u;\xi] &= \mathcal{L}\left[\frac{\Gamma_2(u+1)}{\Gamma_2(u-w+1)}t^{u-w};\xi\right]\\ &= L\left[\frac{\Gamma(u+1)}{\Gamma(u-w_1+1)}t^{u-w_1};\xi_1\right]e_1 + L\left[\frac{\Gamma(u+1)}{\Gamma(u-w_2+1)}t^{u-w_2};\xi_2\right]e_2\\ &= \left[\frac{\Gamma(u+1)}{\xi_1^{u-w_1+1}}\right]e_1 + \left[\frac{\Gamma(u+1)}{\xi_2^{u-w_2+1}}\right]e_2\\ &= \frac{\Gamma_2(u+1)}{\xi^{u-w+1}},\end{aligned}$$

where $u = \varrho_1 + j\varrho_2 = u_1 e_1 + u_2 e_2 \in \mathbb{C}_2$ such that $\mathrm{Re}(\varrho_1) > |\mathrm{Im}(\varrho_2)| - 1$, $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$, and $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$. By applying a similar process, we can find the bicomplex Laplace transform of the bicomplex Riemann-Liouville fractional differentiation of other bicomplex-valued elementary functions of real variable. For $\mathrm{Re}(z_3) > |\mathrm{Im}(z_4)|$, we have

(i) $\boldsymbol{f(t) = e^{at}}$

$$\mathcal{L}\{D^w e^{at}\} = \mathcal{L}\{E_t(-w, a)\}$$

$$= \frac{1}{\xi^{-w}(\xi - a)};\ a \in \mathbb{C}_2,\ \ \mathrm{Re}(z_1) < |\mathrm{Im}(z_2)|,$$

where $E_t(-w, a) = t^{-w}\sum_{k=0}^{\infty} \frac{(at)^k}{\Gamma_2(-w+k+1)}$.

(ii) $\boldsymbol{f(t) = \sin at}$

$$\mathcal{L}[D^w \sin at\,;\ \xi] = \mathcal{L}[S_t(-w, a);\ \xi]$$

$$= \frac{1}{\xi^{-w}(\xi^2 + a^2)};\ a \in \mathbb{C}_2,\ \mathrm{Re}(z_1) < |\mathrm{Im}(z_2)|,$$

where $S_t(-w, a) = t^{-w}\sum_{k_{\text{odd}}}^{\infty} \frac{(-1)^{(k-1)/2}(at)^k}{\Gamma_2(-w+k+1)}$.

(iii) $\boldsymbol{f(t) = \cos at}$

$$\mathcal{L}[D^w \cos at\,; \xi] = \mathcal{L}[C_t(-w, a); \xi]$$

$$= \frac{1}{\xi^{-w-1}(\xi^2 + a^2)};\ a \in \mathbb{C}_2,\ \mathrm{Re}(z_1) < |\mathrm{Im}(z_2)|,$$

where $C_t(-w, a) = t^{-w}\sum_{k_{\text{even}}}^{\infty} \frac{(-1)^{k/2}(at)^k}{\Gamma_2(-w+k+1)}$.

(iv) $\boldsymbol{f(t) = t^{u-1}e^{at}}$

$$\mathcal{L}[D^w t^{u-1} e^{at}; \xi] = \frac{\Gamma_2(u)}{\xi^{-w}(\xi - a)^u};\ a \in \mathbb{C}_2, \mathrm{Re}(z_1) < |\mathrm{Im}(z_2)|,\ \mathrm{Re}(\varrho_1) > |\mathrm{Im}(\varrho_2)|.$$

Thus, we have established the bicomplex Laplace transform of Riemann-Liouville integral and differential operator of bicomplex order and seen that the bicomplex Laplace transform (23) of the bicomplex fractional derivative is more complicated expression than the corresponding formula (17).

Next, we give a result which provide an important tool while dealing with fractional differential equations. Before proceed further, let us introduced the bicomplex Mittag-Leffler function [7] of two parameters as

$$\mathfrak{ML}_{\alpha,\beta}(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{\Gamma_2(\alpha n + \beta)},$$

where $\alpha = z_5 + jz_6 \in \mathbb{C}_2, \beta = z_7 + jz_8 \in \mathbb{C}_2$ with $\mathrm{Re}(z_5) > |\mathrm{Im}(z_6)|$ and $\mathrm{Re}(z_7) > |\mathrm{Im}(z_8)|$, respectively, and $z_p = x_p + iy_p \in \mathbb{C}(i)$. Since, from [31, p.21] for $\mathrm{Re}(z_3) > |a|^{\frac{1}{\alpha_1}} - \mathrm{Im}(z_4)$

$$\int_0^{\infty} e^{-\xi_1 t} t^{\alpha_1 n+\beta_1-1} \mathfrak{ML}_{\alpha_1,\beta_1}^{(n)}(\pm a t^{\alpha_1}) dt = \frac{n! \xi_1^{\alpha_1-\beta_1}}{(\xi_1^{\alpha_1} \mp a)^{n+1}} \tag{24}$$

and for $\mathrm{Re}(z_3) > |a|^{\frac{1}{\alpha_1}} + \mathrm{Im}(z_4)$

$$\int_0^{\infty} e^{-\xi_2 t} t^{\alpha_2 n+\beta_2-1} \mathfrak{ML}_{\alpha_2,\beta_2}^{(n)}(\pm a t^{\alpha_2}) dt = \frac{n!\, \xi_2^{\alpha_2-\beta_2}}{\left(\xi_1^{\alpha_2} \mp a\right)^{n+1}} \ . \tag{25}$$

Therefore, for $\mathrm{Re}(z_3) > |a|^{\frac{1}{\alpha_1}} + |\mathrm{Im}(z_4)|$ a pair of the bicomplex Laplace transforms of the function $t^{\alpha_2 n+\beta_2-1}\mathfrak{ML}_{\alpha_2,\beta_2}^{(n)}(\pm a t^{\alpha_2})$, can be obtained by combining (24) and (25) as the idempotent component and we get

$$\int_0^{\infty} e^{-\xi t} t^{\alpha n+\beta-1} \mathfrak{ML}_{\alpha,\beta}^{(n)}(\pm a t^{\alpha}) dt = \frac{n! \xi^{\alpha-\beta}}{(\xi_1^{\alpha} \mp a)^{n+1}}. \tag{26}$$

4. Application

Differential equations of fractional order appear more and more frequently in various research areas and engineering applications [25-28]. An effective and easy-to-use method for solving such equations is needed. The problem of finding such method becomes more essentials when we come to deal the fractional differential equations of bicomplex order. However, there are some methods for solving fractional differential equation which are limited to rational order [9, 30] or real order [34].

In this section, we introduce a method suitable for a wide class of initial value problem for fractional differential equations. The method uses the bicomplex Laplace transform technique and is based on the formula of the bicomplex Laplace transform of the bicomplex Mittag-Leffler function in two parameters. We hope that this method could be useful for obtaining solutions of different applied problems appearing in physics, chemistry, electrochemistry, engineering, etc. in bicomplex sense.

Let us consider the following initial value problem for a non-homogeneous fractional differential equation under non-zero initial conditions:

$$D^w f(t) - \lambda f(t) = g(t),\ (t > 0) \tag{27}$$

$$[D^{n-m+w} f(t)]_{t=0} = \mathfrak{c}_n,\ (n = 1,2, \dots, m-1), \tag{28}$$

where $m_1 = \lfloor \mathrm{Re}(w_1) \rfloor + 1, m_2 = \lfloor \mathrm{Re}(w_2) \rfloor + 1,$ and $m = \max\{m_1, m_2\}$. On applying Laplace transform to (27) and taking into account the initial conditions (28), we have

$$\xi^w \mathcal{F}(\xi) - \sum_{n=0}^{m-1} \xi^{m-n-1} \mathfrak{c}_n - \lambda \mathcal{F}(\xi) = \mathcal{G}(\xi)$$

$$\Rightarrow \mathcal{F}(\xi) = \sum_{n=0}^{m-1} \frac{\xi^{m-n-1}}{\xi^w - \lambda} \mathfrak{c}_n + \frac{\mathcal{G}(\xi)}{\xi^w - \lambda}. \tag{29}$$

The bicomplex inverse Laplace transform of (29) using (26) gives the solution as follows $f(t) = \frac{1}{2\pi i} \int_\gamma e^{\xi t} \mathcal{F}(\xi) d\xi$ (where γ is bicomplex Bromwich closed contour)

$$= \frac{1}{2\pi i} \sum_{n=0}^{m-1} \mathfrak{c}_n t^{w-m+n} \mathfrak{M}\mathfrak{L}_{w,w-m+n+1}(\lambda t^w) + \frac{1}{2\pi i} \int_0^t (t - \tau)^{w-1} \mathfrak{M}\mathfrak{L}_{w,w}(\lambda (t-\tau)^w) g(\tau) f(\tau)$$

which is the required solution of the fractional differential equation of bicomplex order.

5. Conclusion

We performed some mathematical analysis to derive the bicomplex Laplace transform of Riemann-Liouville operators of bicomplex order. We obtained the bicomplex Laplace transform of Riemann-Liouville integration and differentiation of bicomplex order of some elementary bicomplex-valued functions of real variables. We discussed the bicomplex Laplace transform of mixed order operator (composition of fractional and integer order integral and derivative). As an application, we attained the solution of initial value problem for a non-homogeneous fractional differential equation of bicomplex order by employing bicomplex Laplace and inverse Laplace transforms.

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