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APPLICATIONS OF WAVELET PACKETS TO EULER-BERNOULLI EQUATION AND FREQUENCY DOMAIN ANALYSIS

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Abstract: In signal and image processing the use of wavelets as tool is vital. In present paper, we give the scaling, wavelet discretization of wave propagation equation. Periodic Wavelet Packets are used to discretize the wave propagation equations. We investigate the reduction of wave equation for Euler-Bernoulli beam frequency domain Analysis and discretized these equations using the periodic wavelet packets. The spectral analysis of wave propagation is related to the solution of the problem in transformed wavelet domain.

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1. Introduction

Engineers, Physicists and Mathematicians all together have developed wavelet analysis over past many decades. Grossmann and Morlet [9], Daubechies [6] and Meyer [14] have contributed a lot to develop the theory of Wavelets. There exist many of the bases functions in wavelets in comparison of Fourier expansion. The existence of many bases functions in wavelets led to some very successful applications within the field of signal processing. Coifman et.al. [3, 4, 5] and Wickerhauser [18] have contributed to develop the theory of wavelet packets which is the generalization of wavelets. Meyer [15] and Nielson [16] have discussed periodic wavelets. Hess-Nielson [10] discussed the resemblance between trigonometric system & periodic wavelet packets. Projection method is one of the most important notion of this theory studied by Barker [1] and Latto et. al.[13]. Kumar et. al. [11] studied the periodic boundary value problems (1-D Helmholtz Equation) by discretization method.

Some differential equations can be solved by transformed methods. The wave propagation in structures is given in Doyle's book [7] and wavelet transform is used to study vibration problem. We use periodic wavelets and wavelet packets transform to solve wave propagation problems. Discrete Fourier Transform (DFT) is used to determine Spectrum and dispersion relation for a generalized system. Such relations provide wave numbers and wave speeds. These parameters play an important role to illustrate wave mechanics significantly and are also essential for Spectral Finite Element (SFE) formulation, see Gopalkrishnan and Mitra [8]. Moreover these parameters specify the nature of Wave Mode.

Next for a propagating mode, the nature of frequency variation of wave numbers gives information whether the mode is non-dispersive i.e. dispersive where the shape changes with propagation. In this section, these parameters are explained using the example of a generalized one-dimensional second and fourth order system.

Definition of Multi-resolution analysis (MRA) is very common and can be seen in many papers including [11]. Periodic scaling functions, wavelets and wavelet packets enable us to investigate such kind of cases.

Definition 1.2: Let $\phi \in L^2(\mathbb{R})$ be the scaling function, then 1-periodic scaling function for $j, l \in \mathrm{Z}$ is defined as

$$\tilde{\phi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n) = 2^{j/2} \sum_{n=-\infty}^{\infty} \phi\left(2^j (x+n) - l\right), \quad x \in \mathbb{R} \tag{1}$$

Let $\psi \in L^2(\mathbb{R})$ be the basic wavelet, then the 1-periodic wavelet is defined as

$$\tilde{\psi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x+n) = 2^{j/2} \sum_{n=-\infty}^{\infty} \psi\left(2^j (x+n) - l\right), \quad x \in \mathbb{R} \tag{2}$$

If $\omega_n \in L^2(\mathbb{R})$ be the basic wavelet packet then the 1-periodic wavelet packet is defined as

$$\tilde{\omega}_{n,j,l}(x) = \sum_{p=-\infty}^{\infty} \omega_{n,j,l}(x+p) = 2^{j/2} \sum_{p=-\infty}^{\infty} \omega_n\left(2^j (x+p) - l\right), \quad x \in \mathbb{R} \tag{3}$$

1.1 Connection Coefficients: The connection coefficients for orthonormal baseshave been studied by Barker [1], Latto et. al.[13], Perrier and Wickerhauser [17] and Kunth [12]. These coefficients are defined as

$$\Gamma^{d_1,d_2}_{j,l,m} = \int_{-\infty}^{\infty} \phi^{(d_1)}_{j,l}(x)\, \phi^{(d_2)}_{j,l}(x)\, dx, \; j, l, m \in \mathbb{R}$$

where d_1 and d_2 are orders of differentiations. The derivatives of connection coefficients have been considered well-defined. By variable transformation $x \leftarrow \left(2^j x - l\right)$, we obtain

$\Gamma_{j,l,m}^{d_1,d_2} = 2^{jd}\int_{-\infty}^{\infty}\phi^{(d_1)}(x)\,\phi^{(d_2)}(x-m+l)\,dx = 2^{jd}\,\Gamma_{0,0,m-l}^{d_1,d_2}$, where $d_1 + d_2 = d$.

The identity $\Gamma_{0,0,n}^{d_1,d_2} = (-1)^{d_1}\,\Gamma_{0,0,n}^{0,d}$ can be obtained by repeated use of integration by parts because the support of scaling functions is compact. Hence

$\Gamma_{j,l,m}^{d_1,d_2} = (-1)^{d_1}\,2^{jd}\,\Gamma_{0,0,m-l}^{0,d}$

1.2 Wavelet Packet Expansion of a function $f \in L^2(\mathbb{R})$**:** If a function $f \in L^2(\mathbb{R})$ then

$$f(x) = \sum_{k=0}^{2^{J_0-1}} c_{J_0,k}\,\tilde{\phi}_{J_0,k}(x) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^j-1} d_{n,l,k}\,\tilde{\omega}_{n,l,k}(x) \tag{4}$$

where $l = j - P,\ P = J_0, J_{0+1}, J_{0+2}..., J$

and $d_{n,l,k}$ the wavelet packet coefficients is defined as

$$d_{n,l,k} = \left\langle f,\,\tilde{\omega}_{n,l,k}\right\rangle = \int_{-\infty}^{\infty} f(x)\tilde{\omega}_{n,l,k}(x)\,dx \tag{5}$$

Now from equation (1.4), we have

$$f(x) = \sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\,\tilde{\phi}_{J_0,l}(x) + \sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\,\tilde{\omega}_{n,l,k}(x) \tag{6}$$

where

$$c_{J_0,l} = \int_{-\infty}^{\infty} f(x)\tilde{\phi}_{J_0,l}(x)dx \tag{7}$$

and

$$d_{n,l,k} = \int_{-\infty}^{\infty} f(x)\tilde{\omega}_{n,l,k}(x)\,dx \tag{8}$$

1.3 Expansion of Periodic Functions: Let $f \in \tilde{V}_J$ and J_0 with $0 \le J_0 \le J$. Thedecomposition

$\tilde{V}_J = \tilde{V}_{J_0} \oplus \left(\bigoplus_{j=J_0}^{J-1}\tilde{W}_j\right)$, which is obtain from $\tilde{V}_J \oplus \tilde{W}_J = \tilde{V}_{J+1}$

splits function *f*into pure periodic scaling function expansion

$$f(x)=\sum_{l=0}^{2^J-1} c_{J,l}\,\tilde{\phi}_{J,l}(x), \quad x\in[0,1] \tag{9}$$

and the periodic wavelet expansion

$$f(x)=\sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\tilde{\phi}_{J_0,l}(x)+\sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\tilde{\psi}_{j,l}(x), \quad x\in[0,1] \tag{10}$$

If $J_0=0$, than the equation (1.10) becomes

$$f(x)=c_{0,0}+\sum_{j=J_0}^{J-1}\sum_{l=0}^{2^j-1} d_{j,l}\,\tilde{\psi}_{j,l}(x) \tag{11}$$

Now periodic wavelet packets expansion be

$$f(x)=\sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\,\tilde{\phi}_{J_0,l}(x)+\sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\,\tilde{\omega}_{n,l,k}(x), \; x\in[0,1] \tag{12}$$

Now we define the periodic extension $\tilde{f}$ of f as

$$\tilde{f}(x)=f(x-\lfloor x\rfloor), \; x\in\mathbb{R} \tag{13}$$

Then 1-periodicity of $\tilde{f}$ can be verified as

$$\tilde{f}(x+1)=f(x+1-(\lfloor x+1\rfloor))=f(x-\lfloor x\rfloor)=\tilde{f}(x), \; x\in\mathbb{R} \tag{14}$$

As $\lfloor x\rfloor$ is an integer, we have $\tilde{\phi}(x-\lfloor x\rfloor)=\tilde{\phi}(x), \tilde{\psi}(x-\lfloor x\rfloor)=\tilde{\psi}(x)$ and $\tilde{\omega}_n(x-\lfloor x\rfloor)=\tilde{\omega}_n(x)$, for $x\in\mathbb{R}$. Equation (14) applying (12) gives

$$\begin{aligned}\tilde{f}(x)=f(x-\lfloor x\rfloor)&=\sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\,\tilde{\phi}_{J_0,l}(x-\lfloor x\rfloor)+\sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\,\tilde{\omega}_{n,l,k}(x-\lfloor x\rfloor)\\ &=\sum_{l=0}^{2^{J_0}-1} c_{J_0,l}\,\tilde{\phi}_{J_0,l}(x)+\sum_{j=J_0}^{J-1}\sum_{n=2^P}^{2^{P+1}-1}\sum_{k=0}^{2^J-1} d_{n,l,k}\,\tilde{\omega}_{n,l,k}(x), \; x\in\mathbb{R}\end{aligned} \tag{15}$$

The coefficients in (9), (10) and (12) are respectively given by

$$c_{j,l}=\int_{-\infty}^{\infty} f(x)\tilde{\phi}_{j,l}(x)\,dx\,;\, d_{j,l}=\int_{-\infty}^{\infty} f(x)\tilde{\psi}_{j,l}(x)\,dx\,;\; d_{n,l,k}=\int_{-\infty}^{\infty} f(x)\tilde{\omega}_{n,l,k}(x)\,dx$$

but in fact these coefficients are same as those of the non-periodic expansions. To prove this we use the fact that $\tilde{f}(x) = f(x),\ x \in [0, 1]$ and we have

$$d_{n,l,k} = \int_0^1 \tilde{f}(x)\tilde{\omega}_{n,l,k}(x)\, dx = \sum_{n=-\infty}^{\infty} \int_0^1 \tilde{f}(x)\omega_{n,l,k}(x+n)\, dx = \int_{-\infty}^{\infty} \tilde{f}(y)\omega_{n,l,k}(y)dy \qquad (16)$$

1.4 Differentiating Matrix with Respect to Scaling Function: Let *f* be a function in $V_J \bigcap C^d(\mathbb{R})$ and $J \in N_0$ differentiating both sides of equation $f(x) = \sum_{l=-\infty}^{\infty} c_{J,l}\, \phi_{J,l}(x)$; $x \in \mathbb{R}$, *d*–times, we obtain

$$f^{(d)}(x) = \sum_{l=-\infty}^{\infty} c_{J,l}\, \phi_{J,l}^{(d)}(x) \ ; \qquad x \in \mathbb{R} \qquad (17)$$

It should be noted that $f^{(d)}(x)$ in general not belong to V_j so project $f^{(d)}$ back onto V_j as

$$\left(P_{V_J} f^{(d)}\right)(x) = \sum_{k=-\infty}^{\infty} c_{J,k}^{(d)}\, \phi_{J,k}(x), \qquad x \in \mathbb{R} \qquad (18)$$

where accordingly, we have

$$c_{J,k}^{(d)} = \int_{-\infty}^{\infty} f^{(d)}(x)\phi_{J,k}(x)dx \qquad (19)$$

Substituting equation (17) in equation (19), we get

$$c_{J,k}^{(d)} = \sum_{l=-\infty}^{\infty} c_{J,l} \int_{-\infty}^{\infty} \phi_{J,k}(x)\phi_{J,l}^{(d)}(x)dx \qquad \Rightarrow c_{J,k}^{(d)} = \sum_{l=-\infty}^{\infty} c_{J,l}\, \Gamma_{J,k,l}^{0,d}$$

$$\Rightarrow\ c_{J,k}^{(d)} = \sum_{l=-\infty}^{\infty} c_{J,l}\, 2^{Jd}\, \Gamma_{l-k}^{d} \qquad \Rightarrow c_{J,k}^{(d)} = \sum_{n=-\infty}^{\infty} c_{J,n+k}\, 2^{Jd}\, \Gamma_n^d, \qquad -\infty < k < \infty$$

We used $\Gamma_{J,l,m}^{d_1 d_2} = (-1)^d\, 2^{Jd}\, \Gamma_{m-l}^{d}$ for the Last equality. Since Γ_n^d is only non-zero for $n \in [2-D, D-2]$, so we define

$$c_{J,k}^{(d)} = \sum_{n=2-D}^{D-2} c_{J,n+k}\, 2^{Jd}\, \Gamma_n^d\ ;\ J, k \in \mathbb{Z} \qquad (20)$$

Recall that if *f* is 1–periodic then

$$c_{J,l} = c_{J,l+P^{2^J}}, \qquad l, P \in \mathbb{Z} \ \text{ and } \ c_{J,k}^{(d)} = c_{J,k+P^{2^J}}^{(d)} \qquad k, P \in \mathbb{Z}$$

Hence it is sufficient to consider 2^J coefficient of either type or equation (20) becomes

$$c_{J,k}^{(d)} = \sum_{n=2-D}^{D-2} c_{J,\langle n+k \rangle_{2^J}} \; 2^{Jd} \Gamma_n^d \quad ; \quad k = 0,1,...,2^J - 1 \tag{21}$$

The matrix form of the above system of equationscan be expressed as follows

$$c^{(d)} = D^{(d)} c \tag{22}$$

where $\left| D^{(d)} \right|_{k \langle n+k \rangle_{2^J}} = 2^{Jd} \Gamma_n^d,\ k = 0,1,...,2^J - 1;\ n = 2 - D......, D - 2,$

and $c^{(d)} = \left[c_{J,0}^{(d)}, c_{J,1}^{(d)}, \cdots, c_{J,2^J-1}^{(d)} \right]$

We will refer the matrix $D^{(d)}$ as the differentiation matrix of order *d*. It can be seen that $D^{(d)}$ is symmetric for *d* is even and skew symmetric for *d* is odd.

2. Reduction of Wave Equations (for Euler-Bernoulli beam)

The fourth order flexural wave equation for an Euler Bernoulli beam in time space is given by

$$EI\ \frac{\partial^4 w}{\partial x^2} + \eta A \frac{\partial w}{\partial t} + \rho A\ \frac{\partial^2 w}{\partial t^2} = 0 \tag{23}$$

where $w(x,t)$ is the transverse displacement, I is the moment of inertia of the cross section, A is the Area of cross section and E is the Young's modules. Flexural waves are more dispersivethan the longitudinal waves i.e., the speed of waves vary with frequencies.

Similar to the approximation of the longitudinal displacement $u(x,t)$ given by

$$u(x,t) = u(x,\tau) = \sum_k u_k(x)\varphi(\tau - k), \quad k \epsilon Z \tag{24}$$

and transverse displacement $w(x,t)$is approximated as

$$w(x,t) = w(x,\tau) = \sum_k w_k(x)\varphi(\tau - k); \quad k \epsilon Z \tag{25}$$

where $w_k(x)$ are the approximation coefficients at a certain location x. Substituting Equation (23) in equation (21) we get

$$EI \sum_k \frac{d^4 w_K}{dx^4} \varphi(\tau - k) + \frac{\eta A}{\Delta t} \sum_k w_k \varphi'(\tau - k) + \frac{\rho A}{\Delta t^2} \sum_k w_k \varphi''(\tau - k) = 0 \tag{26}$$

The inner product of equation (26) with $(\tau - j), j = 0,1,2,...,n-1$, yields

$$EI \sum_k \frac{d^4 w_k}{dx^4} \int \varphi(\tau - k)\varphi(\tau - j) d\tau + \frac{\eta A}{\Delta t} \sum_k w_k \int \varphi'(\tau - k)\varphi(\tau - j) d\tau + \frac{\rho A}{\Delta t^2} \sum_k w_k \int \varphi''(\tau - k)\ \varphi(\tau - j) d\tau = 0 \tag{27}$$

The translates of scaling functions mustbe orthogonal i.e.

$$\int \varphi(\tau-k)\varphi(\tau-j)d\tau = 0 \qquad \text{for } j \neq k \tag{28}$$

Using equation (28), equation (27) can be written as

$$EI\,\frac{d^4 w_j}{dx^4} + \frac{\eta A}{\Delta t}\sum_{k=j-N+2}^{j+N-2} \Omega^1_{j-k}\, w_k + \frac{\rho A}{\Delta t^2}\sum_{k=j-N+2}^{j+N-2} \Omega^2_{j-k}\, w_k = 0 \;, j = 0,1,2,\dots,n-1$$

$$\Rightarrow EI\,\frac{d^4 w_j}{dx^4} + \sum_{k=j-N+2}^{j+N-2} \left(\frac{\eta A}{\Delta t}\Omega^1_{j-k} + \frac{\rho A}{\Delta t^2}\Omega^2_{j-k}\right) w_k = 0 \;, j = 0,1,2,\dots,n-1 \tag{29}$$

where N is the order of Daubechies wavelet and Ω^1_{j-k} and Ω^2_{j-k} are the connection coefficient given by

$$\Omega^1_{j-k} = \int \varphi'(\tau-k)\varphi(\tau-j)d\tau \text{ and } \Omega^2_{j-k} = \int \varphi''(\tau-k)\varphi(\tau-j)d\tau$$

For compactly supported wavelets of Daubechies, Ω^1_{j-k} and Ω^2_{j-k} are non-zero only in the interval $k = j-N+2$ to $k = j+N-2$ The details of evaluation of connection coefficients for different order are given by Beylkin [2].

The differential equation (23) under forced boundary conditionsis

$$EI\,\frac{\partial^2 w}{\partial x^2} = M \tag{30}$$

$$EI\,\frac{\partial^3 w}{\partial x^3} = -V \tag{31}$$

where M and V are the applied moment and transverse force respectively and $M(x,t)$ and $V(x,t)$ are given by

$$M(x,t) = M(x,\tau) = \sum_k M_k(x)\varphi(\tau-k)\,, k \epsilon Z \tag{32}$$

$$V(x,t) = V(x,\tau) = \sum_k V_k(x)\varphi(\tau-k)\,,\ k \epsilon Z \tag{33}$$

By equation (30) and (32) we obtain the following ODEs

$$EI\,\frac{d^2 w_j}{dx^2} = M_j,\ j = 0,1,2,\dots,n-1. \tag{34}$$

Similarly by equation (31) and (33)

$$EI\,\frac{d^3 w_j}{dx^3} = -V_j\,,\ j = 0,1,2,\dots,n-1. \tag{35}$$

Spectral element for beam is formulated using the ODEs given by Equations (29), (24) and (35).

Next the coefficients which are not in finite boundaries can be dealt with wavelet extrapolating technique. Thus the ODEs (29) in matrix equation is written as

$$\left\{\frac{d^4 w_j}{dx^4}\right\} + \left(\frac{\eta A}{EI}\Gamma^1 + \frac{\rho A}{EI}\Gamma^2\right)\{w_j\} = 0 \tag{36}$$

Again here the second order connection coefficient matrix Γ^2 is replaced by $[\Gamma^1]^2$. This modification results in the following equation from equation (36)

$$\left\{\frac{d^4 w_j}{dx^4}\right\} + \left(\frac{\eta A}{EI}\Gamma^1 + \frac{\rho A}{EI}[\Gamma^1]^2\right)\left\{ w_j\right\} = 0 \tag{37}$$

The decoupling of Equation (36) is done through eigenvalues analysis and the decoupled equation is obtained as

$$\frac{d^4\widehat{w}_j}{dx^4} + \left(\frac{\eta A}{EI}\gamma_j + \frac{\rho A}{EI}\gamma_j^2\right)\widehat{w}_j = 0, j = 0,1,2,\dots,n-1 \tag{38}$$

where

$$\widehat{w}_j = \varphi^{-1}w_j \tag{39}$$

φ is the eigenvector matrix and $i\gamma_j$ are the eigenvalues of the matrix Γ^1 as given by the equations

$$\Gamma^1 = \varphi\pi\varphi^{-1} \tag{40}$$

and $$\Gamma^2 = \varphi\pi^2\varphi^{-1} \tag{41}$$

The forced boundary conditions given by equation (34) and (35) are similarly transformed as

$$EI \frac{d^2\widehat{w}_j}{dx^2} = \widehat{M}_j \ , j = 0,1,2,\dots,n-1 \tag{42}$$

$$EI \frac{d^3\widehat{w}_j}{dx^3} = -\widehat{V}_j, j = 0,1,2,\dots,n-1 \tag{43}$$

Now we discretized the above equation as follows.

The equation (42) can be written as

$$\left.\begin{aligned} EI\, \widehat{w}_j''(x) &= \widehat{M}_j\ (x) \\ \widehat{w}_j(x) &= \widehat{w}_j(x+1) \end{aligned}\right\} x \epsilon\ \mathbb{R} \tag{44}$$

where $EI\epsilon\ \mathbb{R}$ and $\widehat{M}_j(x) = \widehat{M}_j(x+1)$.

2.1. Representation with respect to scaling functions: The discretization of equation (44) by $\widehat{w}_j(x)$replaced by the approximation

$$\widehat{w}_{jJ}(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\widehat{w}_j}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\}, \quad J\epsilon\ N_o \tag{45}$$

$$\widehat{w}_{jJ}''(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\widehat{w}_j}^{(2)}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} \tag{46}$$

where $\left(C_{\widehat{w}_j}^{(2)}\right)_{Jk}$ are defined by

$$\left(C_{\widehat{w}_j}^{(2)}\right)_{Jk} = \left[D^{(2)}C_{\widehat{w}_j}\right]_{Jk} = \sum_{n=2-D}^{D-2}[C_{\widehat{w}_j}]_{J(n+k)2^J}2^{Jd}\,\Gamma_n^d, k = 0,1,2,\dots,2^J-1 \tag{47}$$

The coefficients $\left[C_{\widehat{w}_j}\right]_{Jk}$ can be evaluated by the Galerkin Method. Multiplying equation (44) by $\tilde{\varphi}_{Jk}(x)$ and and integrating over the limit [0,1], we get

$$EI\int_0^1 \widehat{w}_{jJ}^{''}(x)\,\tilde{\varphi}_{Jk}(x)dx = \int_0^1 \widehat{M}_j(x)\,\tilde{\varphi}_{Jk}(x)dx$$

Using equations (45), (46) and orthogonality of periodized scaling functions, we get

$$EI\left(C_{\widehat{w}_j}^{(2)}\right)_{Jk} = \left(C_{\widehat{M}_j}\right)_{Jk},\; k = 0,1,2,3\ldots,2^J-1 \tag{48}$$

where

$$[C_{\widehat{M}_j}]_{Jk} = \int_0^1 \widehat{M}_j(x)\,\tilde{\varphi}_{Jk}(x)dx \tag{49}$$

In vector notation this becomes

$$EIC_{\widehat{w}_j}^{(2)} = C_{\widehat{M}_j} \tag{50}$$

We obtain following system of linear equations

$$AC_{\widehat{w}_j} = C_{\widehat{M}_j} \tag{51}$$

where

$$A = EID^{(2)} \tag{52}$$

Alternatively we can replace $D^{(2)}$ by D^2, where D is given by the equation $D = D^{(1)}$ and obtain

$$AC_{\widehat{w}_j} = C_{\widehat{M}_j} \tag{53}$$

where

$$A = EID^2 \tag{54}$$

Equation (51) and (53) represents the scaling function discretization of equation (44)

2.2. Representation with respect to Wavelets: Taking equation (53) as a point of departure and using the relations$d_{\widehat{w}_j} = WC_{\widehat{w}_j}$ and$d_{\widehat{M}_j} = WC_{\widehat{M}_j}$, we get

$$AW^Td_{\widehat{w}_j} = W^TC_{\widehat{M}_j} \tag{55}$$

Let us consider $\breve{A} = WAW^T$, then $\breve{A} = W(EID^{(2)})W^T \Rightarrow \breve{A} = EI\breve{D}^{(2)}$ (56)

where $\breve{D}^{(2)}$ is defined as in following equation $D^{(2)} = WD^{(2)}W^T$.

Then from equation (55), we get

$$WAW^Td_{\widehat{w}_j} = WW^TC_{\widehat{M}_j} \Rightarrow \breve{A}d_{\widehat{w}_j} = C_{\widehat{M}_j} \tag{57}$$

This is the wavelet discretization of equation (44).

2.3. Representation in terms of Wavelet packets: Taking equation (51) as a point of departure and using the relations $d_{\widehat{w}_j} = \omega_n C_{w_j}$ and $d_{\widehat{M}_j} = \omega_n C_{\widehat{M}_j}$, we get

$$A\omega_n^T d_{\widehat{w}_j} = \omega_n^T d_{\widehat{M}_j} \tag{58}$$

Let us define $A^* = \omega_n A \omega_n^T$, then

$$A^* = \omega_n \left(EID^{(2)}\right)\omega_n^T \quad \text{or} \quad A^* = EID^{*(2)} \tag{59}$$

where $D^{*(2)}$ is defined as in following equation $D^{*(2)} = \omega_n D^{(2)} \omega_n^T$.

Then from equation (58), we get

$$\omega_n A \omega_n^T d_{\widehat{w}_j} = \omega_n \omega_n^T d_{\widehat{M}_j} \Rightarrow A^* d_{\widehat{w}_j} = d_{\widehat{M}_j} \tag{60}$$

This is the wavelet packet discretization of equation (42).

Also, the equation (41) can be written as

$$\left.\begin{aligned} -EI\, \widehat{w}_j'''(x) &= \widehat{V}_j\ (x) \\ w_j(x) &= w_j(x+1) \end{aligned}\right\} x \in \mathbb{R} \tag{61}$$

where $EI \in \mathbb{R}$ and $\widehat{V}_j(x) = \widehat{V}_j(x+1)$.

2.4. Representation in terms of scaling functions: The discretization of equation (61) by $\widehat{w}_j(x)$ replaced by the approximation

$$\widehat{w}_{jJ}(x) = \sum_{k=0}^{2^J-1} \left\{ \left(C_{\widehat{w}_j}\right)_{Jk} \tilde{\varphi}_{Jk}(x) \right\},\ J \in N_o \tag{62}$$

we find that

$$\widehat{w}_{jJ}'''(x) = \sum_{k=0}^{2^J-1} \left\{ \left(C_{\widehat{w}_j}^{(3)}\right)_{Jk} \tilde{\varphi}_{Jk}(x) \right\} \tag{63}$$

where $\left(C_{\widehat{w}_j}^{(3)}\right)_{Jk}$ are defined by

$$\left(C_{\widehat{w}_j}^{(3)}\right)_{Jk} = \left[D^{(3)} C_{\widehat{w}_j}\right]_{Jk} = \sum_{n=2-D}^{D-2} [C_{\widehat{w}_j}]_{J(n+k)2^J} 2^{Jd}\, \Gamma_n^d, k = 0,1,2,\dots,2^J-1 \tag{64}$$

We can determine coefficients $\left[C_{\widehat{w}_j}\right]_{Jk}$ as in equation (47). Multiplying equation (61) by $\tilde{\varphi}_{Jk}(x)$ and integrating over the limit $[0,1]$, we get

$$-EI \int_0^1 \widehat{w}_{jJ}'''(x)\, \tilde{\varphi}_{Jk}(x) dx = \int_0^1 \widehat{V}_j\ (x)\, \tilde{\varphi}_{Jk}(x) dx$$

Using equations (62), (63) and orthogonality of periodized scaling functions, we get

$$-EI \left(C_{\widehat{w}_j}^{(3)}\right)_{Jk} = \left(C_{\widehat{V}_j}\ \right)_{Jk} \quad K = 0,1,2,3\dots,2^J-1 \tag{65}$$

where

$$[C_{\hat{V}_j}]_{JK} = \int_0^1 \hat{V}_j \ (x)\, \tilde{\varphi}_{Jk}(x) dx \tag{66}$$

In vector notation this becomes

$$-EIC_{\hat{w}_j}^{(3)} = C_{\hat{V}_j} \tag{67}$$

we get the linear system of equations

$$GC_{\hat{w}_j} = C_{\hat{V}_j} \tag{68}$$

where

$$G = -EID^{(3)} \tag{69}$$

Alternatively we can replace $D^{(3)}$ by D^3, where D is given by the equation $D = D^{(1)}$ and obtain

$$GC_{\hat{w}_j} = C_{\hat{V}_j} \tag{70}$$

where

$$G = -EID^3 \tag{71}$$

Equations (66), (70) represent the scaling function discretization of equation (61).

2.5. Representation in terms of Wavelets: Taking equation (66) as a point of departure and using the relations $d_{\hat{w}_j} = W \ C_{\hat{w}_j}$ and $d_{\hat{V}_j} = W \ C_{\hat{V}_j}$, we get

$$GW^T d_{\hat{w}_j} = W^T d_{\hat{V}_j} \tag{72}$$

Let us consider $\breve{G} = WGW^T$, then

$$\breve{G} = W(-EID^{(3)})W^T \text{or } \breve{G} = -EI\breve{D}^{(3)} \tag{73}$$

Where $\breve{D}^{(3)}$ is defined as in following equation $\breve{D}^{(3)} = WD^{(3)}W^T$.

Then from equation (72), we get

$$WGW^T d\hat{w}_j = WW^T d_{\hat{V}_j} \Rightarrow \breve{G} d_{\hat{w}_j} = d_{\hat{V}_j} \tag{74}$$

This is the wavelet discretization of equation (61).

2.6. Representation with respect to Wavelet packets: Taking equation (68) as a point of departure and using the relation $d_{\hat{w}_j} = \omega_n C_{\hat{w}_j}$ and $d_{\hat{V}_j} = \omega_n C_{\hat{V}_j}$, we get

$$G\omega_n^T d_{\hat{w}_j} = \omega_n^T d_{\hat{V}_j} \tag{75}$$

Let us define $G^* = \omega_n G \omega_n^T$, then

$$G^* = \omega_n\left(-EID^{(3)}\right)\omega_n^T \text{or } G^* = -EID^{*(3)} \tag{76}$$

Where $D^{*(3)}$ is defined as in following equation $D^{*(3)} = \omega_n D^{(3)} \omega_n^T$.

Then from equation (75), we get

$$\omega_n G \omega_n^T d_{\widehat{w}_j} = \omega_n \omega_n^T d_{\widehat{V}_j} \Rightarrow G^* d_{\widehat{w}_j} = d_{\widehat{V}_j} \quad (77)$$

This is the wavelet packet discretization of equation (61).

3. Frequency Domain Analysis

We have investigated non-periodic wavelet spectral finite elements (WSFE) formulation in context of time domain analysis of wave propagation. There the boundaries are dealt with wavelet extrapolation technique. Apart from such non-periodic formulation, WSFE can also be formulated assuming periodicity of the solution.

Though the Latter Method is not capable of accurately simulating the time domain wave response it helps in deriving the frequency domain wave properties. In thecurrent approach function $u(x,t)$has been taken periodic in time. Considering the discretized $u(x,\tau)$ to be periodic with time period t_f in the equation

$$EA\,\frac{d^2 u_j}{dx^2} = \sum_{k=j-N+2}^{j+N-2} \left(\frac{\eta A}{\Delta t}\Omega_{j-k}^1 + \frac{\rho A}{\Delta t^2}\Omega_{j-k}^2\right) u_k, j = 0,1,2,\dots,n-1 \quad (78)$$

The unknown coefficient u_j on L.H.S. are taken as

$$\left.\begin{array}{l} u_{-1} = u_{n-1} \\ u_{-2} = u_{n-2} \\ \dots\dots\dots\dots \\ u_{-N+2} = u_{n-N+2} \end{array}\right\} \quad (79)$$

Similarly the unknown coefficient on R.H.S. i.e. u_n , $u_{n+1},\dots,u_{n+N-2}$ are equal to u_0 , $u_1,\dots,u_{N-2}$ respectively. Under these assumptions equation (78) reduce in the following form

$$\left\{\frac{d^2 u_j}{dx^2}\right\} = \left(\frac{\eta A}{EA}\,\Delta^1 + \frac{\rho A}{EA}\Delta^2\right)\{u_j\} \quad (80)$$

Where Δ^1 and Δ^2 are n× n circulant connection coefficient matrices and have the form

$$\Delta^1 = \frac{1}{\Delta t}\begin{bmatrix} \Omega_0^1 & \Omega_{-1}^1 & \dots\Omega_{-N+2}^1\dots\Omega_{N-2}^1 & \dots & \Omega_1^1 \\ \Omega_1^1 & \Omega_0^1 & \dots\Omega_{-N+3}^1\dots 0 & \dots & \Omega_2^1 \\ \dots & \dots & \dots\dots\quad\dots\dots & \dots & \dots \\ \dots & \dots & \dots\dots\quad\dots\dots & \dots & \dots \\ \Omega_{-1}^1 & \Omega_{-2}^1 & \dots 0\ \dots\Omega_{N-3}^1 & \dots & \Omega_0^1 \end{bmatrix}$$

Δ^2 for second order derivative has a similar form for a circulant matrix Δ^1 the eigen values α_j are

$$\alpha_j = \sum_{k=-N+2}^{N-2} \left(\Omega_k^1 e^{-2\pi ijk/n}\right) \quad j = 0,1,2,3,\dots,n-1 \quad (81)$$

and the corresponding orthogonal Eigen vectors $v_j,\ j = 0,1,2,3,\ldots,n-1$ are

$$(v_j)_k = \frac{1}{\sqrt{n}} e^{-2\pi ijk/n},\ k = 0,1,2,3,\ldots,n-1 \tag{82}$$

For Δ^1 , $\Omega_p^1 = -\Omega_{-p}^1$ for $p = 0,1,2,3,\ldots,N-2$ and $\Omega_0^1 = 0$ and we can write $\alpha_i = i\lambda_j$ with

$$\lambda_j = \frac{-2}{\Delta t}\sum_{k=1}^{N-2} \quad \Omega_k^1\, Sin\left(\frac{2\pi kj}{n}\right), j = 0,1,2,3,\ldots,n-1 \tag{83}$$

Again here, through the second order connection coefficient matrix Δ^2 can be evaluated independently, they can also be written as

$$\Delta^2 = [\Delta^1]^2 \tag{84}$$

Thus equation (3.3) can be written as

$$\left\{\frac{d^2 u_j}{dx^2}\right\} = \left(\frac{\eta A}{EA}\ \Delta^1 + \frac{\rho A}{EA}[\Delta^1]^2\right)\{u_j\} \tag{85}$$

We have already discussed the spectral element formulation includingeigenvalues analysis. It is used to diagonalize the matrix in equation (80) and decouple the ODEs. The eigenvaluesα_j and eigenvectors $(v_j)_k$can be analytically determined for the periodic boundary conditions which optimize the computational cost. Thus the diagonalized form of equation (85) is

$$\left\{\frac{d^2 \hat{u}_j}{dx^2}\right\} = \left(\frac{\eta A}{EA}\lambda_j + \frac{\rho A}{EA}\lambda_j^2\right)\{\hat{u}_j\}\ ,\ j = 0,1,2,3,\ldots,n-1 \tag{86}$$

where

$$\hat{u}_j = \varphi^{-1}\, u_j \tag{87}$$

Here φ being the eigenvector matrix, Neglecting damping equation (86) can be written as

$$\frac{d^2 \hat{u}_j}{dx^2} = -\frac{\rho A}{EA}\lambda_j^2 \hat{u}_j \quad , j = 0,1,2,3,\ldots,n-1 \tag{88}$$

The matrix equation

$$\begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ \ldots.. \\ u_{n-1} \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \varphi_{N-2} & \ldots\ldots\ldots\varphi_2 & \varphi_1 \\ \varphi_1 & 0 & 0 & 0 & \ldots\ldots\ldots\varphi_3 & \varphi_2 \\ & \varphi_2\varphi_1 & 0 & 0 & \ldots\ldots\ldots.\varphi_4 & \varphi_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ & \varphi_{N-2}\varphi_{N-3}\varphi_{N-4} & \ldots & & \ldots\ldots\ldots 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots\ldots\ldots\ldots & \ldots \\ 0 & 0 & 0 & \varphi_{N-3} & \ldots\ldots\ldots\ \varphi_1 & 0 \end{bmatrix} \begin{Bmatrix} u_0 \\ u_1 \\ u_2 \\ \ldots \\ u_{n-1} \end{Bmatrix} \tag{89}$$

where u_j , $\varphi_j$are the values of $u(x,\tau)$ and $\varphi(\tau) at\ \tau = j$ is known as the periodic sultion of the wavelet transformation equation

$$u(x,t) = u(x,\tau) = \sum_k u_k(x)\varphi(\tau - k)\ k\epsilon\, Z \tag{90}$$

For such circulant matrix the equation (89) can be replaced by following convolution relation

$$\{\widehat{U}_j\} = \{\widehat{K}\varphi_j, \hat{u}_j\} \tag{91}$$

Where $\{\widehat{U}_j\}, \{\hat{u}_j\}$ are FFT of $\{U_j\}$ *and* $\{u_j\}$ respectively. $\{\widehat{K}\varphi_j\}$ is fast Fourier Transform (FFT) of $\{K\varphi\} = \{0, \varphi_1, \varphi_2, \ldots, \varphi_{N-2}, \ldots, 0\}$,which is the first column of the scaling function matrix given by equation (89). Similarly in equation (85) the matrix Δ^1 is also a circulant matrix and thus it can be written as (neglecting the damping).

$$\left\{\frac{d^2\hat{u}_j}{dx^2}\right\} = \frac{\rho A}{EA}\{\widehat{K}^2_{\Omega j}, \hat{u}_j\} \tag{92}$$

where $\{\widehat{K}_{\Omega j}\}$ are the FFT of $K_\Omega = \{\Omega^1_0, \Omega^1_{-1}, \Omega^1_{-N+2}, \ldots, \Omega^1_{N-2}, \ldots, \Omega^1_1\}$, which is the first column of the connection coefficient matrix Δ^1 substituting equation (91) in equation (92), we get

$$\left\{\frac{d^2}{dx^2}\left(\frac{\widehat{U}_j}{\widehat{K}_{\varphi j}}\right)\right\} = \frac{\rho A}{EA}\left\{\widehat{K}^2_{\Omega j}, \left(\frac{\widehat{U}_j}{\widehat{K}_{\varphi j}}\right)\right\} \tag{93}$$

$$or \qquad \left\{\frac{d^2\widehat{U}_j}{dx^2}\right\} = \frac{\rho A}{EA}\{\widehat{K}^2_{\Omega j}, \widehat{U}_j\} \tag{94}$$

It can be easily seen that the FFT coefficient$\widehat{K}_{\Omega j}$ are equal to the eigenvalues$i\lambda_j$ of the matrix Δ^1 given by equation (83) Thus equation (94) can be written as

$$\frac{d^2\widehat{U}_j}{dx^2} = \frac{-\rho A}{EA}\lambda_j^2\widehat{U}_j, j = 0, 1, 2, 3, \ldots, n-1 \tag{95}$$

It should be mentioned here that by relating the equation (88) and (95). It can be observed that the transformation given by equation (87) is similar to DFT for periodic WSFE formulation.

In Fourier transform-based spectral Analysis, the transformed ODEs are given by

$$\frac{d^2\widehat{U}_j}{dx^2} = \frac{-\rho A}{EA}\omega_j^2\widehat{U}_j, \; j = 0, 1, 2, 3, \ldots, n-1 \tag{96}$$

where

$$\omega_j = \frac{2\pi j}{n\Delta t} (3.20)$$

It is noticed that for a given sampling rate Δt, λ_j exactly matches ω_j upto a certain frequency P_N of Nyguist frequency $f_{nyq} = \frac{1}{2\Delta t}$ and equation (96) can be discretized as.

Now equation (96) can be written as

$$\left.\begin{array}{c}\widehat{U}_j''(x) + \frac{\rho A}{EA}\omega_j^2\widehat{U}_j = 0\\ \widehat{U}_j(x) = \widehat{U}_j(x+1)\end{array}\right\}; \text{ where} \frac{\rho A}{EA} \in \mathbb{R}, \quad x \in \mathbb{R} \tag{97}$$

3.1. Representation in terms ofscaling functions: The discretization of equation (97) by $\widehat{U}_j\ (x)$ replaced by the approximation

$$\widehat{U}_{jJ}(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\widehat{u}_j}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} \quad J\epsilon\ N_o \tag{98}$$

we find

$$\widehat{U}_{jJ}''(x) = \sum_{k=0}^{2^J-1}\left\{\left(C_{\widehat{u}_j}^{(2)}\right)_{Jk}\tilde{\varphi}_{Jk}(x)\right\} \tag{99}$$

where $\left(C_{\widehat{u}_j}^{(2)}\right)_{Jk}$ are defined by

$$\left(C_{\widehat{u}_j}^{(2)}\right)_{Jk} = \left[D^{(2)}C_{\widehat{u}_j}\right]_{Jk} = \sum_{n=2-D}^{D-2}\left[C_{\widehat{u}_j}\right]_{J(n+k)2^J} 2^{Jd}\,\Gamma_n^d, k = 0,1,2,\ldots,2^J-1 \tag{100}$$

We can use the Galerkin Method to calculate the coefficients $\left[C_{\widehat{u}_j}\right]_{Jk}$. Multiplying equation (97) by $\tilde{\varphi}_{Jk}(x)$and integrating over the limit $[0,1]$ we get

$$\int_0^1 \widehat{U}_{jJ}''(x)\,\tilde{\varphi}_{Jk}(x)dx + \frac{\rho A\,\omega_j^2}{EA}\int_0^1 \widehat{U}_j\ (x)\,\tilde{\varphi}_{Jk}(x)dx = 0$$

Using equation (98), (99) and orthogonality of periodized scaling functions we get

$$\left(C_{\widehat{u}_j}^{(2)}\right)_{Jk} + \frac{\rho A\,\omega_j^2}{EA}\left(C_{\widehat{u}_j}\ \right)_{Jk} = 0,\ k = 0,1,2,3,\ldots,2^J-1 \tag{101}$$

In vector notation this becomes

$$C_{\widehat{u}_j}^{(2)} + \frac{\rho A\,\omega_j^2}{EA}C_{\widehat{u}_j} = 0 \tag{102}$$

We get the following linear system of equations

$$AC_{\widehat{u}_j} = 0 \tag{103}$$

where

$$A = D^{(2)} + \frac{\rho A\,\omega_j^2}{EA}I \tag{104}$$

Alternatively we can replace $D^{(2)}$ by D^2 Where D is given by the equation $D^{(1)} = D$ and obtain

$$AC_{\widehat{u}_j} = 0 \tag{105}$$

where

$$A = D^2 + \frac{\rho A\,\omega_j^2}{EA}I \tag{106}$$

Equations (101), (103) represent the scaling function discretization of equation (95).

3.2. Representation in terms of Wavelets: Taking equation (101) as a point of departure and using the relation$d_{\hat{u}_j} = WC_{\hat{u}_j}$, we get

$$AW^T d_{\hat{u}_j} = 0 \tag{107}$$

Let us define $\breve{A} = WAW^T$, then

$$\breve{A} = W\left(D^{(2)} + \frac{\rho A\, \omega_j^2}{EA} I \right)W^T \ \ or \qquad \breve{A} = \breve{D}^{(2)} + \frac{\rho A\, \omega_j^2}{EA} \mathrm{I} \tag{108}$$

where $\breve{D}^{(2)}$ is defined as infollowing equation $\breve{D}^{(2)} = WD^{(2)}W^T$.

Then from equation (107), we get

$$WAW^T d_{\hat{u}_j} = 0 \quad \Rightarrow \breve{A} d_{\hat{u}_j} = 0 \tag{109}$$

This is the wavelet discretization of equation (97).

3.3. Representation in terms of Wavelet packets: Taking equation (103) as a point of departure and using the relation $d_{\hat{u}_j} = \omega_n C_{\hat{u}_j}$, we get

$$A\omega_n^T d_{\hat{u}_j} = 0 \tag{110}$$

Let us define $A^* = \omega_n A \omega_n^T$, then

$$A^* = \omega_n \left(D^{(2)} + \frac{\rho A\, \omega_j^2}{EA} I \right)\omega_n^T \ \ or \qquad A^* = D^{*(2)} + \frac{\rho A\, \omega_j^2}{EA} \mathrm{I} \tag{111}$$

where $D^{*(2)}$ is defined as in following equation $D^{*(2)} = \omega_n D^{(2)} \omega_n^T$.

Then from equation (3.34) we get

$$\omega_n A \omega_n^T d_{\hat{u}_j} = 0 \quad \Rightarrow A^* d_{\hat{u}_j} = 0 \tag{112}$$

The wavelet packet discretization of equation (97) is given by (112).

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