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FRACTIONAL INTEGRAL OPERATORS INVOLVINGMITTAG-LEFFLER TYPE *E*-FUNCTION

Sanjay Bhatter[*]and **Sheikh Mohammed Faisal**[**]
Email: *bhatters@gmail.com **sheikhmohammedf9@gmail.com
Department of Mathematics, Malaviya National Institute of Technology
Jaipur 302017, India

Abstract: In this paper, we define a pair of multidimensional fractional integral operators whose kernelinvolvegeneralized multivariable polynomial $S_V^{U_1,\ldots,U_k}(x_1,\ldots,x_k)$ and Mittag-Leffler type *E*-function. First we obtain images of useful functions, then we establish results concerning Mellin transform, Mellin convolutions and inversion formulae for these operators.

Key Words: Fractional integral operator, $\overline{H}-$function, General class of multivariable polynomials, Mittag-Leffler type *E*-function.

AMS Subject Classification Code: 26A33, 33E20, 33E12.

1. Introduction

The multivariable polynomial $S_V^{U_1,\ldots,U_k}(x_1,\ldots,x_k)$ introduced by Srivastava and Garg [8] is defined in the following manner:

$$S_V^{U_1,\ldots,U_k}\left[x_1,\ldots,x_k\right]=\sum_{R_1,\ldots,R_k=0}^{\sum_{i=1}^{k}U_iR_i\le V}(-V)_{\sum_{i=1}^{k}U_iR_i}\,A(V,R_1,\ldots,R_k)\,\frac{x_i^{R_i}}{R_i\,!}\quad ,V=0,1,2,\ldots \tag{1}$$

where $U_1,\ldots,U_k$ are arbitrary positive integers and the coefficients $A(V,R_1,\ldots,R_k)$ are arbitrary constants(real or complex).

The Mittag-Leffler type *E*-function is defined by Bhatter and Faisal [1] as follows

$${}_{\tau}E_k^h\left[Z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right.\right]={}_{\tau}E_k^h\left[Z\left|\begin{matrix}(\rho,a);(\gamma_1,q_1;s_1),\ldots,(\gamma_h,q_h;s_h)\\(\alpha,\beta);(\delta_1,p_1;r_1),\ldots,(\delta_k,p_k;r_k)\end{matrix}\right.\right]$$

$$=\sum_{n=0}^{\infty}\frac{\left[(\gamma_1)_{q_1 n}\right]^{s_1}\cdots\left[(\gamma_h)_{q_h n}\right]^{s_h}(-1)^{\rho n}Z^{an+\tau}}{\left[(\delta_1)_{p_1 n}\right]^{r_1}\cdots\left[(\delta_k)_{p_k n}\right]^{r_k}\Gamma(\alpha n+\beta)} \tag{2}$$

where

$$\left.\begin{array}{l} Z,\alpha,\beta,\gamma_i,\delta_j\in C;\ \Re(\alpha)\ge 0,\Re(\beta)>0,\Re(\gamma_i)>0,\Re(\delta_j)>0,q_i\ge 0,\\ p_j\ge 0;\ s_i\ge 0,r_j\ge 0;\ a,\tau\in R;\ \rho\in\{0,1\},\left(\sum_{i=1}^{h}q_i s_i<\sum_{j=1}^{k}p_j r_j+\Re(\alpha)\right)\ or\\ \left(\sum_{i=1}^{h}q_i s_i=\sum_{j=1}^{k}p_j r_j+\Re(\alpha)\ when\ \prod_{i=1}^{h}(q_i)^{q_i s_i}\left[\alpha^{\alpha}\prod_{i=1}^{k}(p_j)^{p_j r_j}\right]^{-1}\left|Z^{a}\right|<1\right)\\ \qquad\qquad for\ \ i=1,2,\ldots,h;\ j=1,2,\ldots,k. \end{array}\right\} \tag{3}$$

The Mellin-Barnes type contour integral representation of M-L type *E*-function [2] is given as follows

$${}_{\tau}E_k^h\left[Z\left|\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right.\right]=\frac{\prod_{v=1}^{k}\left[\Gamma(\delta_v)\right]^{r_v}}{\prod_{u=1}^{h}\left[\Gamma(\gamma_u)\right]^{s_u}}\frac{Z^{\tau}}{2\pi i}\int_{\wp}\frac{\Gamma(\zeta)\Gamma(1-\zeta)\prod_{i=1}^{h}\left[\Gamma(\gamma_i-q_i\zeta)\right]^{s_i}}{\Gamma(\beta-\alpha\zeta)\prod_{j=1}^{k}\left[\Gamma(\delta_j-p_j\zeta)\right]^{r_j}}\left[(-1)^{\rho}(-Z^{a})\right]^{-\zeta}d\zeta \tag{4}$$

where $\wp$ is a suitable contour of integration that runs from $C-i\infty$ to $C+i\infty,\ C\in\mathbb{R}$ and intended

to separate the poles of the integrand at $\zeta=-n$ for all $n\in\mathbb{N}_0$ (to the left) from those at $\zeta=n+1$

and at $\zeta=\dfrac{\gamma_i+n}{q_i}$ i=1,2,...,h; for all $n\in\mathbb{N}_0$ (to the right).

The $\overline{H}$ -function [5] is defined and represented in the following manner:

$$\overline{H}_{P,Q}^{M,N}[Z]=\overline{H}_{P,Q}^{M,N}\left(Z\left|\begin{matrix}(\varepsilon_j,\omega_j;A_j)_{1,N},(\varepsilon_j,\omega_j)_{N+1,P}\\(b_j,\vartheta_j)_{1,M},(b_j,\vartheta_j;B_j)_{M+1,Q}\end{matrix}\right.\right)=\frac{1}{2\pi i}\int_{L}\overline{\phi}(\xi)Z^{\xi}d\xi\ \ (z\ne 0) \tag{5}$$

where $\overline{\phi}(\xi)=\dfrac{\prod_{j=1}^{M}\Gamma(b_j-\vartheta_j\xi)\prod_{j=1}^{N}\{\Gamma(1-\varepsilon_j+\omega_j\xi)\}^{A_j}}{\prod_{j=M+1}^{Q}\{\Gamma(1-b_j+\vartheta_j\xi)\}^{B_j}\prod_{j=N+1}^{P}\Gamma(\varepsilon_j-\omega_j\xi)}$ (6)

It may be noted that $\bar{\phi}(\xi)$ contains fractional powers of some of the gamma functions. M,N,P,Qare integers such that $1 \le M \le Q$, $0 \le N \le P$, $(\omega_j)_{1,P}$, $(\vartheta_j)_{1,Q}$ are positive real numbers and $(A_j)_{1,N}$, $(B_j)_{M+1,Q}$ may take non-integer values, which we assume to be positive for standardization purpose. $(\varepsilon_j)_{1,P}$ and $(b_j)_{1,Q}$ are complex numbers such that the points $\xi = \dfrac{b_j + k}{\vartheta_j},\ j = 1,\ldots,M;\ k = 0,1,2,\ldots;$ which are the poles of $\Gamma\left(b_j - \vartheta_j\xi\right),\ j = 1,\ldots,M$; and the points $\xi = \dfrac{\varepsilon_j - 1 - k}{\omega_j},\ j = 1,\ldots,N;\ k = 0,1,2,\ldots;$ which are the singularities of $\left\{\Gamma\left(1 - \varepsilon_j + \omega_j\xi\right)\right\}^{A_j},\ j = 1,\ldots,N$; do not coincide. The contour L is the line from $c - i\infty$ to $c + i\infty,\ c \in$ □ suitably intended to keep the poles of $\Gamma\left(b_j - \vartheta_j\xi\right),\ j = 1,\ldots,M$; to the right of the path, and the singularities of $\left\{\Gamma\left(1 - \varepsilon_j + \omega_j\xi\right)\right\}^{A_j},\ j = 1,\ldots,N$; to the left of the path.

The following sufficient conditions for the absolute convergence of the defining integral for $\overline{H}$ - function given by (5) have been given by Gupta et al. [5]:

$$\left.\begin{array}{l}
\text{(i) } \left|\arg(Z)\right| < 1/2\,\Omega\pi \text{ and } \Omega > 0 \\[2ex]
\text{(ii) } \left|\arg(Z)\right| = 1/2\,\Omega\pi \text{ and } \Omega \ge 0 \\[2ex]
\text{and (a) } \mu \ne 0 \text{ and the contour } L \text{ is so chosen that } \left(c\mu + \lambda + 1\right) < 0 \\[2ex]
\text{(b) } \mu = 0 \textit{ and } \left(\lambda + 1\right) < 0
\end{array}\right\} \tag{7}$$

where

$$\Omega = \sum_{1}^{M}\vartheta_j + \sum_{1}^{N}\omega_j A_j - \sum_{M+1}^{Q}\vartheta_j B_j - \sum_{N+1}^{P}\omega_j \tag{8}$$

$$\mu = \sum_{1}^{N}\omega_j A_j + \sum_{N+1}^{P}\omega_j - \sum_{1}^{M}\vartheta_j - \sum_{M+1}^{Q}\vartheta_j B_j \tag{9}$$

$$\lambda = \operatorname{Re}\left(\sum_{1}^{M} b_j + \sum_{M+1}^{Q} b_j B_j - \sum_{1}^{N}\varepsilon_j A_j - \sum_{N+1}^{P}\varepsilon_j\right) + \frac{1}{2}\left(-M - \sum_{M+1}^{Q} B_j + \sum_{1}^{N} A_j + P - N\right) \tag{10}$$

It may be noted that the conditions of validity given above are more general than those given earlier [3].

The following series representation of the $\overline{H}$ -function was given by Rathie [6]:

$$\overline{H}_{P,Q}^{M,N}\left(Z \left| \begin{matrix} (\varepsilon_j,\omega_j;A_j)_{1,N},(\varepsilon_j,\omega_j)_{N+1,P} \\ (b_j,\vartheta_j)_{1,M},(b_j,\vartheta_j;B_j)_{M+1,Q} \end{matrix} \right. \right) = \sum_{\upsilon=1}^{M}\sum_{t=0}^{\infty}\overline{\theta}\left(S_{t,\upsilon}\right)Z^{S_{t,\upsilon}} \tag{11}$$

$$\text{where}\,\overline{\theta}(S_{t,\upsilon}) = \frac{\prod\limits_{j=1,\,j\neq\upsilon}^{M}\Gamma(b_j-\vartheta_jS_{t,\upsilon})\prod\limits_{j=1}^{N}\{\Gamma(1-\varepsilon_j+\omega_jS_{t,\upsilon})\}^{A_j}\left(-1\right)^t}{\prod\limits_{j=M+1}^{Q}\{\Gamma(1-b_j+\vartheta_jS_{t,\upsilon})\}^{B_j}\prod\limits_{j=N+1}^{P}\Gamma(\varepsilon_j-\omega_jS_{t,\upsilon})t!\,\vartheta_\upsilon}, \quad S_{t,\upsilon} = \frac{b_\upsilon+t}{\vartheta_\upsilon} \tag{12}$$

To the sequel, we shall also, make use of the following behavior of the $\overline{H}$ -function for small and large value of z as recorded by Saxena [7]

$$\overline{H}_{P,Q}^{M,N}\left[Z\right] = O\left[\left|Z\right|^{\Delta}\right] \text{for small z, where } \Delta = \min_{1\le j\le M}\operatorname{Re}\left(\frac{b_j}{\vartheta_j}\right) \tag{13}$$

$$\overline{H}_{P,Q}^{M,N}\left[Z\right] = O\left[\left|Z\right|^{\nabla}\right] \text{for large z, where } \nabla = \max_{1\le j\le N}\left[\operatorname{Re}\left\{A_j\left(\frac{\varepsilon_j-1}{\omega_j}\right)\right\}\right] \tag{14}$$

provided that either of the following conditions are satisfied:

$$\left.\begin{matrix} (i) \ \ \kappa<0 \text{ and } 0<\left|Z\right|<\infty \\ (ii) \ \ \kappa=0 \text{ and } 0<\left|Z\right|<K^{-1} \end{matrix}\right\} \tag{15}$$

$$\left.\begin{matrix} \text{where (i)} \quad \kappa = \sum\limits_{1}^{N}\omega_jA_j + \sum\limits_{N+1}^{P}\omega_j - \sum\limits_{1}^{M}\vartheta_j - \sum\limits_{M+1}^{Q}\vartheta_jB_j \\ \text{(ii)}\,K = \prod\limits_{1}^{N}\left(\omega_j\right)^{\omega_jA_j}\prod\limits_{N+1}^{P}\left(\omega_j\right)^{\omega_j}\prod\limits_{1}^{M}\left(\vartheta_j\right)^{-\vartheta_j}\prod\limits_{M+1}^{Q}\left(\vartheta_j\right)^{-\vartheta_jB_j} \end{matrix}\right\} \tag{16}$$

Throughout the paper we assume that

$$f\left(t_1,...,t_s\right) = \begin{cases} O\prod\limits_{j=1}^{s}\left(\left|t_j\right|^{\psi_j}\right) & \max\left\{\left|t_j\right|\right\}\to 0 \\ O\prod\limits_{j=1}^{s}\left(\left|t_j\right|^{-\zeta_j}e^{-W_j\left|t_j\right|}\right) & \min\left\{\left|t_j\right|\right\}\to\infty \end{cases} \qquad j=1,...,s \tag{17}$$

Such a class of function will be represented symbolically as $f\left(t_1,...,t_s\right)\in \mathrm{A}$.

We also assume that $\int\ldots\int_{\Omega_s}\left|f\left(t_1,\ldots,t_s\right)\right|dt_1\ldots dt_s<\infty$ for every bounded s-dimensional region Ω_s except the origin.

2. **Multidimensional Fractional Integral Operators**

In the present paper we study the following fractional integral operators

$$I_x\left[f\left(t_1,\ldots,t_s\right)\right]=I_{x;U,V;Z}^{\Lambda,\sigma;e,f;\eta,\lambda}\left[f\left(t_1,\ldots,t_s\right);x_1,\ldots,x_s\right]$$

$$=\left(\prod_{j=1}^{s}x_j^{-\Lambda_j-\sigma_j}\right)\int_0^{x_1}\ldots\int_0^{x_s}\left[\prod_{j=1}^{s}t_j^{\Lambda_j}\left(x_j-t_j\right)^{\sigma_j-1}\right]S_V^{U_1,\ldots,U_s}\left[E_1\left(\frac{t_1}{x_1}\right)^{e_1}\left(1-\frac{t_1}{x_1}\right)^{f_1},\ldots,E_s\left(\frac{t_s}{x_s}\right)^{e_s}\left(1-\frac{t_s}{x_s}\right)^{f_s}\right]$$

$$\times{}_\tau E_k^h\left[Z\prod_{j=1}^{s}\left(\frac{t_j}{x_j}\right)^{\eta_j}\left(1-\frac{t_j}{x_j}\right)^{\lambda_j}\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right]f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s \qquad (18)$$

where

(i) $\min\operatorname{Re}(e_j,f_j,\eta_j a,\lambda_j a)\geq 0$ andall parameters $e_j,f_j,\eta_j a,\lambda_j a$ are not zero simultaneously ($j=1,\ldots,s$).

(ii) $\min\operatorname{Re}\left[1+\Lambda_j+\eta_j\tau+\psi_j\right]>0,\ \min\operatorname{Re}\left[\sigma_j+\lambda_j\tau\right]>0\ \left(j=1,\ldots,s\right)$

$$J_x\left[f\left(t_1,\ldots,t_s\right)\right]=J_{x;U,V;Z}^{\Lambda,\sigma;e,f;\eta,\lambda}\left[f\left(t_1,\ldots,t_s\right);x_1,\ldots,x_s\right]$$

$$=\left(\prod_{j=1}^{s}x_j^{\Lambda_j}\right)\int_{x_1}^{\infty}\ldots\int_{x_s}^{\infty}\left[\prod_{j=1}^{s}t_j^{-\Lambda_j-\sigma_j}\left(t_j-x_j\right)^{\sigma_j-1}\right]S_V^{U_1,\ldots,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{e_1}\left(1-\frac{x_1}{t_1}\right)^{f_1},\ldots,E_s\left(\frac{x_s}{t_s}\right)^{e_s}\left(1-\frac{x_s}{t_s}\right)^{f_s}\right]$$

$$\times{}_\tau E_k^h\left[Z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{\eta_j}\left(1-\frac{x_j}{t_j}\right)^{\lambda_j}\,\middle|\,\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right]f\left(t_1,\ldots,t_s\right)dt_1\ldots dt_s \qquad (19)$$

where

(i) $\min\operatorname{Re}(e_j,f_j,\eta_j a,\lambda_j a)\geq 0$ andall parameters $e_j,f_j,\eta_j a,\lambda_j a$ are not zero simultaneously ($j=1,\ldots,s$).

(ii) $\operatorname{Re}\left(W_j\right)=0,\min\operatorname{Re}\left[\Lambda_j+\eta_j\tau+\zeta_j\right]>0$ or

$\operatorname{Re}\left(W_j\right)>0,\min\operatorname{Re}\left[\sigma_j+\lambda_j\tau\right]>0\ \left(j=1,\ldots,s\right)$.

3. **Some Useful Images**

Now we find the images of some useful functions in our operators of study

a)

$$I_x\left[\prod_{j=1}^{s} t_j^{\upsilon_j}\left(h_j+t_j\right)^{-\varphi_j}\right]=Z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\sum_{R_1,\ldots,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}\left(-V\right)_{\sum_{j=1}^{s}U_jR_j}A\left(V,R_1,\ldots,R_s\right)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}$$

$$\times\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(n+1\right)}\prod_{j=1}^{s}\left(-\frac{x_j}{h_j}\right)^{n}\left(\frac{x_j^{\upsilon_j}}{h_j^{\varphi_j}}\right)\left(1+\frac{x_j}{h_j}\right)^{\sigma_j+f_jR_j+\lambda_j\tau-\varphi_j}$$

$$\times\overline{H}{}^{1,h+3s+1}_{h+3s+1,k+2s+2}\left[\left(-1\right)^{\rho}\left(-Z^{a}\right)\left(1+\frac{x_j}{h_j}\right)^{\lambda_j a}\left|\begin{array}{l}(1-\gamma_i,q_i;s_i)_{1,h},\left(-\Lambda_j-\upsilon_j-e_jR_j-\eta_j\tau,\eta_j a;1\right)_{1,s},\left(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_j a;1\right)_{1,s},\\ \left(\varphi_j-\sigma_j-\Lambda_j-\upsilon_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(0,1;1);—\\ (0,1);(1-\delta_j,p_j;r_j)_{1,k},\left(\varphi_j-\sigma_j-\Lambda_j-\upsilon_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},\\ \left(-\sigma_j-\Lambda_j-\upsilon_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(1-\beta,\alpha;1)\end{array}\right.\right] \tag{20}$$

provided that $\min\operatorname{Re}(e_j,f_j,\eta_j a,\lambda_j a)\ge 0$,all parameters $e_j,f_j,\eta_j a,\lambda_j a$ are not zero simultaneously, $\min\operatorname{Re}\left[1+\Lambda_j+\eta_j\tau+\upsilon_j\right]>0$ and $\min\operatorname{Re}\left[\sigma_j+\lambda_j\tau\right]>0$ ($j=1,\ldots,s$).

b)

$$J_x\left[\prod_{j=1}^{s} t_j^{\upsilon_j}\left(h_j+t_j\right)^{-\varphi_j}\right]=Z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\sum_{R_1,\ldots,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}\left(-V\right)_{\sum_{j=1}^{s}U_jR_j}A\left(V,R_1,\ldots,R_s\right)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}$$

$$\times\sum_{n=0}^{\infty}\frac{1}{\Gamma\left(n+1\right)}\prod_{j=1}^{s}\left(x_j^{\upsilon_j-\varphi_j}\right)\left(-\frac{h_j}{x_j}\right)^{n}\left(1+\frac{h_j}{x_j}\right)^{\sigma_j+f_jR_j+\lambda_j\tau-\varphi_j}$$

$$\times\overline{H}^{1,h+3s+1}_{h+3s+1,k+2s+2}\left[(-1)^{\rho}\left(-Z^{a}\right)\left(1+\frac{h_j}{x_j}\right)^{\lambda_j a}\left|\begin{matrix}(1-\gamma_i,q_i;s_i)_{1,h},\left(1-\Lambda_j+\upsilon_j-e_jR_j-\eta_j\tau-\varphi_j,\eta_j a;1\right)_{1,s},\left(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_j a;1\right)_{1,s},\\ \left(1-\sigma_j-\Lambda_j+\upsilon_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(0,1;1);—\\ (0,1);(1-\delta_j,p_j;r_j)_{1,k},\left(1-\sigma_j-\Lambda_j+\upsilon_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},\\ \left(1-\sigma_j-\Lambda_j+\upsilon_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(1-\beta,\alpha;1)\end{matrix}\right.\right]$$

(21)

provided that $\min\operatorname{Re}(e_j, f_j, \eta_j a, \lambda_j a)\geq 0$, all parameters $e_j, f_j, \eta_j a, \lambda_j a$ are not zero simultaneously, $\min\operatorname{Re}\left[\Lambda_j+\eta_j\tau-\upsilon_j+\varphi_j\right]>0$ and $\min\operatorname{Re}\left[\sigma_j+\lambda_j\tau\right]>0$ ($j=1,...,s$).

Proof: To prove (20), we express the *I*-operator involved in its LHS in the integral form with the help of equation (18). Next, we express generalized multivariable polynomial $S_V^{U_1,...,U_s}\left(x_1,...,x_s\right)$ occurring therein in the series by using (1). Then, we change the order of the series and t_j-integrals and express theMittag-Leffler type *E*-function in terms of Mellin Barnes type contour integrals with the help of(4). Now change the order of ξ and t_j-integrals$\left(j=1,...,s\right)$ (which is permissible under the given conditions). Finally, evaluating the t_j-integrals with the help of known result [4, p.287, Eq. 3.197(8)] we get

$$I_x\left[\prod_{j=1}^{s}t_j^{\nu_j}\left(h_j+t_j\right)^{-\varphi_j}\right]=Z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\leq V}(-V)_{\sum_{j=1}^{s}U_jR_j}A\left(V,R_1,...,R_s\right)\frac{E_1^{R_1}}{R_1!}\cdots\frac{E_s^{R_s}}{R_s!}\prod_{j=1}^{s}\left(x_j^{\nu_j-\varphi_j}\right)$$

$$\times\left(\frac{h_j}{x_j}\right)^{-\varphi_j}\frac{1}{2\pi i}\int_L\overline{\theta}(\xi)(-1)^{\rho\xi}\left(-Z^{a}\right)^{\xi}B\left(\sigma_j+f_jR_j+\lambda_j a\xi+\lambda_j\tau,\Lambda_j+\nu_j+e_jR_j+\eta_j a\xi+\eta_j\tau+1\right)$$

$$\times{}_2F_1\left[\begin{matrix}\varphi_j,\Lambda_j+\nu_j+e_jR_j+\eta_j\tau+\eta_j a\xi+1\\ \sigma_j+\Lambda_j+\nu_j+\left(f_j+e_j\right)R_j+\left(\lambda_j+\eta_j\right)\tau+\left(\lambda_j+\eta_j\right)a\xi+1\end{matrix};\left(-\frac{x_j}{h_j}\right)\right]d\xi \qquad (22)$$

where

$$\left|\arg\left(\frac{x_j}{h_j}\right)\right|<\pi,\ \operatorname{Re}\left(\sigma_j+\lambda_j\tau+f_jR_j+\lambda_j a\xi\right)>0,\ \operatorname{Re}\left(\Lambda_j+\eta_j\tau+\nu_j+e_jR_j+\eta_j a\xi+1\right)>0$$

$j=1,...,s$

Now reinterpreting the result thus obtained in terms of the $\overline{H}$ -function, we easily arrive at the desired result after a little simplification.

Again the proof of result (19) is similar to that of result I.

4. Multidimensional Generalized Stieltjes Transform and Fractional Integral Operators

The multidimensional generalized Stieltjes transform of a function $\phi(\mathrm{t}_1, \mathrm{t}_2, ..., \mathrm{t}_\mathrm{s})$ is defined as

$$S_{w_1,...,w_s}(\phi)(g_1,...,g_s) = \int_0^\infty ... \int_0^\infty \phi(t_1,...,t_s) \prod_{j=1}^{s} \left\{ (t_j + g_j)^{-w_j} \right\} dt_1...dt_s \tag{23}$$

provided that the integral exists.

The following theorem gives the multidimensional generalized Stieltjes transform of the generalized fractional operators given by (18) and (19).

Theorem 1. Let $\phi(\mathrm{t}_1, \mathrm{t}_2, ..., \mathrm{t}_\mathrm{s}) \in A$, then

a) $$S_{w_1,...,w_s}(I_t\phi)(g_1,...,g_s) = \int_0^\infty ... \int_0^\infty \phi(x_1,...,x_s)\psi_1(x_1,...,x_s;g_1,...,g_s)\, dx_1...dx_s \tag{24}$$

b) $$S_{w_1,...,w_s}(J_t\phi)(g_1,...,g_s) = \int_0^\infty ... \int_0^\infty \phi(x_1,...,x_s)\psi_2(x_1,...,x_s;g_1,...,g_s)\, dx_1...dx_s \tag{25}$$

where

$$\psi_1(x_1,...,x_s;g_1,...,g_s) = J_x\left[\prod_{j=1}^{s} (g_j + t_j)^{-w_j} \right]$$

$$= Z^\tau \frac{\prod_{m=1}^{k} [\Gamma(\delta_m)]^{r_m}}{\prod_{l=1}^{h} [\Gamma(\gamma_l)]^{s_l}} \sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s} U_j R_j \le V} (-V)_{\sum_{j=1}^{s} U_j R_j} A(V, R_1, ..., R_s) \frac{E_1^{R_1}}{R_1!} ... \frac{E_s^{R_s}}{R_s!}$$

$$\times \sum_{n=0}^{\infty} \frac{1}{\Gamma(n+1)} \prod_{j=1}^{s} \left(x_j^{-w_j} \right) \left(-\frac{g_j}{x_j} \right)^n \left(1 + \frac{g_j}{x_j} \right)^{\sigma_j + f_j R_j + \lambda_j \tau - w_j}$$

$$\times\overline{H}^{1,h+3s+1}_{h+3s+1,k+2s+2}\left[(-1)^{\rho}\left(-Z^{a}\right)\left(1+\frac{g_j}{x_j}\right)^{\lambda_j a}\left|\begin{array}{l}(1-\gamma_i,q_i;s_i)_{1,h},\left(1-\Lambda_j-e_jR_j-\eta_j\tau-w_j,\eta_j a;1\right)_{1,s},\left(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_j a;1\right)_{1,s},\\ \left(1-\sigma_j-\Lambda_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(0,1;1);—\\ (0,1);(1-\delta_j,p_j;r_j)_{1,k},\left(1-\sigma_j-\Lambda_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},\\ \left(1-\sigma_j-\Lambda_j-w_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(1-\beta,\alpha;1)\end{array}\right.\right] \tag{26}$$

providedthat $\min\operatorname{Re}(e_j,\, f_j,\eta_j a,\lambda_j a)\geq 0$, all parameters $e_j,\, f_j,\eta_j a,\lambda_j a$ are not zero simultaneously, $\min\ \operatorname{Re}\left[\Lambda_j+\eta_j\tau+w_j\right]>0$ and $\min\operatorname{Re}\left[\sigma_j+\lambda_j\tau\right]>0$ ($j=1,\dots,s$).

and $\psi_2\left(x_1,\dots,x_s;g_1,\dots,g_s\right)=I_x\left[\prod_{j=1}^{s}\left(g_j+t_j\right)^{-w_j}\right]$

$$=Z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\sum_{R_1,\dots,R_s=0}^{\sum_{j=1}^{s}U_jR_j\leq V}(-V)_{\sum_{j=1}^{s}U_jR_j}\,A\left(V,R_1,\dots,R_s\right)\frac{E_1^{\,R_1}}{R_1!}\cdots\frac{E_s^{\,R_s}}{R_s!}$$

$$\times\sum_{n=0}^{\infty}\frac{1}{\Gamma(n+1)}\prod_{j=1}^{s}\left(g_j^{-w_j}\right)\left(-\frac{x_j}{g_j}\right)^{n}\left(1+\frac{x_j}{g_j}\right)^{\sigma_j+f_jR_j+\lambda_j\tau-w_j}$$

$$\times\overline{H}^{1,h+3s+1}_{h+3s+1,k+2s+2}\left[(-1)^{\rho}\left(-Z^{a}\right)\left(1+\frac{x_j}{g_j}\right)^{\lambda_j a}\left|\begin{array}{l}(1-\gamma_i,q_i;s_i)_{1,h},\left(-\Lambda_j-e_jR_j-\eta_j\tau,\eta_j a;1\right)_{1,s},\left(1-\sigma_j-f_jR_j-\lambda_j\tau-n,\lambda_j a;1\right)_{1,s},\\ \left(w_j-\sigma_j-\Lambda_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(0,1;1);—\\ (0,1);(1-\delta_j,p_j;r_j)_{1,k},\left(w_j-\sigma_j-\Lambda_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},\\ \left(-\sigma_j-\Lambda_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau-n,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s},(1-\beta,\alpha;1)\end{array}\right.\right] \tag{27}$$

provided that $\min\operatorname{Re}(e_j,\, f_j,\eta_j a,\lambda_j a)\geq 0$ and all parameters $e_j,\, f_j,\eta_j a,\lambda_j a$ are not zero simultaneously, $\min\ \operatorname{Re}\left[1+\Lambda_j+\eta_j\tau\right]>0$ and $\min\operatorname{Re}\left[\sigma_j+\lambda_j\tau\right]>0$ ($j=1,\dots,s$).

It is assumed that the integrals on the RHS of equations (24) and (25) exist.

Proof: To prove first part of Theorem 1, we express the LHS of (24) with the help of (12) and (23), then we interchange the order of t_j and x_j integrals (which is permissible under the conditions stated with the theorem). Finally evaluating the inner t_j-integrals with the help of result (20) (taking ν_j=0 therein), we arrive at desired result after a little simplification. Similarly the result (25) of Theorem 1 can be established on using (21).

Now, the following theorem gives the fractional integrals of generalized Stieltjes transform given by (23).

Theorem 2. Let $\phi(\mathrm{t}_1, \mathrm{t}_2, ..., \mathrm{t}_\mathrm{s}) \in A$, $\min \mathrm{Re}(e_j, f_j, \eta_j a, \lambda_j a) \geq 0$ and all parameters $e_j, f_j, \eta_j a, \lambda_j a$ are not zero simultaneously and $\min \mathrm{Re}\left[\sigma_j + \lambda_j \tau\right] > 0$ ($j = 1, ..., s$), then

a) For $\min\ \mathrm{Re}\left[1 + \Lambda_j + \eta_j \tau\right] > 0$ (j=1,2,...,s)

$$I_y\left[S_{w_1,...,w_s}\phi(t_1,...,t_s)(x_1,...,x_s)\right] = \int_0^\infty ... \int_0^\infty \phi(t_1,...,t_s)\psi_2(t_1,...,t_s;x_1,...,x_s)\,dt_1...dt_s \tag{28}$$

b) For $\min\ \mathrm{Re}\left[\Lambda_j + \eta_j \tau + w_j\right] > 0$ (j=1,2,...,s)

$$J_y\left[S_{w_1,...,w_s}\phi(t_1,...,t_s)(x_1,...,x_s)\right] = \int_0^\infty ... \int_0^\infty \phi(t_1,...,t_s)\psi_1(t_1,...,t_s;x_1,...,x_s)\,dt_1...dt_s \tag{29}$$

where $\psi_1(t_1,...,t_s;x_1,...,x_s)$ and $\psi_2(t_1,...,t_s;x_1,...,x_s)$ are as given in (26) and (27)respectively, provided that the integrals in the RHS of equations (28) and (29) exist.

Proof: Results (28) and (29) of Theorem 2 can be obtained on the similar lines to the proof of Theorem 1. Also we can easily obtain the one dimensional analogues of the Theorem 1 and 2.

5. Mellin Transforms

The multidimensional Mellin transform of the function $f(t_1,...,t_s) \in \mathrm{A}$ is defined by the following equation [9, part I, p.125, Eq. (3.5)].

$$M\left[f(t_1,...,t_s);\theta_1,...,\theta_s\right] = \int_0^\infty ... \int_0^\infty \left(\prod_{j=1}^{s} t_j^{\theta_j - 1}\right) f(t_1,...,t_s)\,dt_1...dt_s \tag{30}$$

provided that the integral exists. Now we shall establish the following results:

Result 1

If $M\left[I_x\left\{f(t_1,...,t_s);\theta_1,...,\theta_s\right\}\right]$ and the conditions of existence of the operator $I_{x;U,V;Z}^{\Lambda,\sigma;e,f;\eta,\lambda}\left[f(t_1,...,t_s)\right]$ exist, then

$$M\left[I_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]=M\left[f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right]\chi\left(\theta_1,...,\theta_s\right) \tag{31}$$

Result 2

If $M\left[I_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]$ and the conditions of existence of the operator $J_{x;U,V;Z}^{\Lambda,\sigma;e,f;\eta,\lambda}\left[f\left(t_1,...,t_s\right)\right]$ exist, then

$$M\left[J_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]=M\left[f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right]\chi\left(1-\theta_1,...,1-\theta_s\right) \tag{32}$$

where

$$\chi\left(\theta_1,...,\theta_s\right)=Z^{\tau}\frac{\prod_{m=1}^{k}\left[\Gamma\left(\delta_m\right)\right]^{r_m}}{\prod_{l=1}^{h}\left[\Gamma\left(\gamma_l\right)\right]^{s_l}}\sum_{R_1,...,R_s=0}^{\sum_{j=1}^{s}U_jR_j\le V}\left(-V\right)_{\sum_{j=1}^{s}U_jR_j}A\left(V,R_1,...,R_s\right)\frac{E_1^{R_1}}{R_1!}...\frac{E_s^{R_s}}{R_s!}$$

$$\times\overline{H}_{h+2s+1,k+s+2}^{1,h+2s+1}\left[\left(-1\right)^{\rho}\left(-Z^{a}\right)\middle|\begin{array}{l}(1-\gamma_i,q_i;s_i)_{1,h},\left(1-\sigma_j-f_jR_j-\lambda_j\tau,\lambda_ja;1\right)_{1,s},(0,1;1),\\ \left(-\Lambda_j+\theta_j-e_jR_j-\eta_j\tau,\eta_ja;1\right)_{1,s};—\\ (0,1);(1-\delta_j,p_j;r_j)_{1,k},(1-\beta,\alpha;1),\\ \left(-\sigma_j-\Lambda_j+\theta_j-\left(e_j+f_j\right)R_j-\left(\lambda_j+\eta_j\right)\tau,\left(\lambda_j+\eta_j\right)a;1\right)_{1,s}\end{array}\right] \tag{33}$$

Proof: To prove result 1, we write the multidimensional Mellin transform of the *I*-operator with the help of equation (30), and then we change the order of t_j and x_j-integrals. Next, with the help of (20) and (30) we arrive at the desired result (31) after simplification. The proof of result 2 can be developed by proceeding on the lines similar to those indicated above.

6. Inversion Formulas

By using the inversion theorems for the multidimensional Mellin transform(30), given by Srivastava and Panda [9, part I, p.125, Lemma 2], the following in version formula for the fractional integral operators defined by (18) and (19)can be obtained as follows

Result 3.

$$f\left(t_1,...,t_s\right)=\frac{1}{\left(2\pi i\right)^{s}}\int_{c_1-i\infty}^{c_1+i\infty}...\int_{c_s-i\infty}^{c_s+i\infty}\frac{\prod_{j=1}^{s}t_j^{-\theta_j}}{\chi\left(\theta_1,...,\theta_s\right)}M\left[I_x\left\{f\left(t_1,...,t_s\right);\theta_1,...,\theta_s\right\}\right]d\theta_1...d\theta_s \tag{34}$$

Result 4.

$$f\left(t_1,\dots,t_s\right)=\frac{1}{\left(2\pi i\right)^s}\int_{c_1-i\infty}^{c_1+i\infty}\dots\int_{c_s-i\infty}^{c_s+i\infty}\frac{\prod_{j=1}^{s}t_1^{-\theta_j}}{\chi\left(1-\theta_1,\dots,1-\theta_s\right)}M\left[J_x\left\{f\left(t_1,\dots,t_s\right);\theta_1,\dots,\theta_s\right\}\right]d\theta_1\dots d\theta_s \tag{35}$$

The precise validity conditions for the inversion formulas (34) and (35) can be deduced from the existence condition of the fractional integral operators defined by (18) and (19) and their multidimensional Mellin transform stated earlier.

7. Mellin Convolutions

The multidimensional Mellinconvolutions of pair of functions $f\left(t_1,\dots,t_s\right)$ and $g\left(t_1,\dots,t_s\right)$ is defined by

$$\left(f*g\right)\left(t_1,\dots,t_s\right)=\left(g*f\right)\left(t_1,\dots,t_s\right)=\int_0^\infty\dots\int_0^\infty\left(\prod_{j=1}^{s}x_j^{-1}\right)f\left(\frac{t_1}{x_1},\dots,\frac{t_s}{x_s}\right)g\left(x_1,\dots,x_s\right)dx_1\dots dx_s \tag{36}$$

provided the multiple integral exists.

If $f\left(t_1,\dots,t_s\right)\in\mathrm{A}$, then the fractional integral operators given by (18) and (19) can be expressed as multidimensional Mellin convolutions as follows:

Result 5.

$$I_{x;U,V;Z}^{\Lambda,\sigma;e,f;\eta,\lambda}g\left(t_1,\dots,t_s\right)=\left(I_{\Lambda,\sigma;e,f;\eta,\lambda;x;U,V;Z}*g\right)\left(x_1,\dots,x_s\right) \tag{37}$$

where $I_{\Lambda,\sigma;e,f;\eta,\lambda;x;U,V;Z}=\left(\prod_{j=1}^{s}x_j^{-\Lambda_j-\sigma_j}\left(x_j-1\right)^{\sigma_j-1}\Theta\left(x_j-1\right)\right)$

$$S_V^{U_1,\dots,U_s}\left[E_1\left(x_1\right)^{-e_1-f_1}\left(x_1-1\right)^{f_1},\dots,E_s\left(x_s\right)^{-e_s-f_s}\left(x_s-1\right)^{f_s}\right]$$

$${}_\tau E_k^h\left[Z\prod_{j=1}^{s}\left(x_j\right)^{-\eta_j-\lambda_j}\left(x_j-1\right)^{\lambda_j}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right.\right] \tag{38}$$

$\Theta\left(x\right)$ being the Heaviside unit function.

Result 6.

$$J^{\Lambda,\sigma;e,f;\eta,\lambda}_{x;U,V;Z}\, g\left(t_1,\dots,t_s\right) = \left(J_{\Lambda,\sigma;e,f;\eta,\lambda;x;U,V;Z} * g\right)\left(x_1,\dots,x_s\right) \tag{39}$$

where

$$J_{\Lambda,\sigma;e,f;\eta,\lambda;x;U,V;Z} = \left(\prod_{j=1}^{s} x_j^{\Lambda_j}\left(1-x_j\right)^{\sigma_j-1}\Theta\left(1-x_j\right)\right)$$

$$\times S_V^{U_1,\dots,U_s}\left[E_1\left(x_1\right)^{e_1}\left(1-x_1\right)^{f_1},\dots,E_s\left(x_s\right)^{e_s}\left(1-x_s\right)^{f_s}\right]$$

$$\times {}_\tau E_k^h\left[z\prod_{j=1}^{s}\left(x_j\right)^{\eta_j}\left(1-x_j\right)^{\lambda_j}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right.\right] \tag{40}$$

$\Theta(x)$ being the Heaviside unit function.

Proof: To prove result 5, we write the *I*-operator defined by (18) in the following form using the Heaviside unit function U(x)

$$I^{\Lambda,\sigma;e,f;\eta,\lambda}_{x;U,V;Z}\, g\left(t_1,\dots,t_s\right) = \int_0^\infty\dots\int_0^\infty\left(\prod_{j=1}^{s} t_j^{-1}\right)\left\{\prod_{j=1}^{s}\left[\left(\frac{x_j}{t_j}\right)^{-\Lambda_j-\sigma_j}\left(\frac{x_j}{t_j}-1\right)^{\sigma_j-1}\Theta\left(\frac{x_j}{t_j}-1\right)\right]\right\}$$

$$\times S_V^{U_1,\dots,U_s}\left[E_1\left(\frac{x_1}{t_1}\right)^{-e_1-f_1}\left(\frac{x_1}{t_1}-1\right)^{f_1},\dots,E_s\left(\frac{x_s}{t_s}\right)^{-e_s-f_s}\left(\frac{x_s}{t_s}-1\right)^{f_s}\right]$$

$$\times {}_\tau E_k^h\left[Z\prod_{j=1}^{s}\left(\frac{x_j}{t_j}\right)^{-\eta_j-\lambda_j}\left(\frac{x_j}{t_j}-1\right)^{\lambda_j}\left|\begin{matrix}(\rho,a);(\gamma_i,q_i;s_i)_{1,h}\\(\alpha,\beta);(\delta_i,p_i;r_i)_{1,k}\end{matrix}\right.\right] g\left(t_1,\dots,t_s\right)dt_1\dots dt_s \tag{41}$$

By using the equation (38) and the definition of the Mellin convolutions given by (36) in the above equation, we arrive at the desired result. The proof of the result 6 can be developed on the same lines.

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