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A GENERAL VOLTERRA-TYPE FRACTIONAL EQUATION ASSOCIATED WITH AN INTEGRAL OPERATOR WITH THE *H*-FUNCTION IN THE KERNEL

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Abstract: In this paper, we solve a general Volterra-type fractional equation associated with an integral operator with the H-function in its Kernel. We make use of convolution technique to solve the main equation. Since H-function occuring in the fractional operator herein is general in nature we can obtain a number of special cases from our main findings, by specializing the parameters of the H-function. We record here two such special cases which involve Fox-Wright function and Mittag-Leffler function respectively. The main result derived in this paper also generalizes a result by Tomovski et al.[8]

Key Words: Convolution Integral Equation; H-function; Laplace transform.

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1. Introduction

The H-function occurring in the present paper is defined and represented in the following manner ([5],p.12)

$$H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_j,\alpha_j)_{1,p}\\ (b_j,\beta_j)_{1,q}\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\cdots,(a_p,\alpha_p)\\ (b_1,\beta_1),\cdots,(b_q,\beta_q)\end{matrix}\right.\right]$$

$$:= \sum_{h=1}^{m}\sum_{r=0}^{\infty}\Theta(\mathfrak{s}_{h,r})(-1)^r z^{\mathfrak{s}_{h,r}} \tag{1}$$

$$\Theta(\mathfrak{s}_{s,r}) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\mathfrak{s}_{h,r})\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\mathfrak{s}_{h,r})}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\mathfrak{s}_{h,r})\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\mathfrak{s}_{h,r})}, \tag{2}$$

and

$$1 \le m \le q \text{ and } 0 \le n \le p\ (m,q \in \mathbb{N} = \{1,2,3,\cdots\};\ n,p \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}), \tag{3}$$

where

$$\mathfrak{s}_{h,r} = (b_h + r)/\beta_h$$

For convergence, existence conditions and other details of the Fox H-function refer the book ([5], p.11-12)

2. An Integral Operator Involving Fox's H-Function

In our present investigation, we make use of the following fractional integral operator with Fox's H-function in its kernel(see, for details, [4]:

$$\left(\mathcal{H}^{w;1,n;\alpha}_{a+;p,q;\beta}\varphi\right)(x) := \int_a^x (x-t)^{\beta-1} H^{1,n}_{p,q}[w(x-t)^{\alpha}]\varphi(t)dt \tag{4}$$

$$(\Re(\beta) > 0;\ \ w \in \mathbb{C}\backslash\{0\};\ \ 0 \le n \le p;\ \ \Re(\beta) + \min_{1\le j\le m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0).$$

The following property of Laplace transform([2], p.13)

$$L(f^{(n)}(x); s) = s^n F(s)$$

holds provided that $f^{(i)}(0) = 0, i = 0,1,2,..,n-1$, n being a positive integer, where

$$L(f(x); s) = \int_0^x e^{-sx} f(x)dx = F(s)$$

The well-known convolution theorem for Laplace transform

$$L\left(\int_0^x f(x-u)g(u)du; s\right) = L(f(x); s)L(g(x); s) \tag{5}$$

holds provided that the various Laplace transforms occuring in (5) exists.

For $a = 0$, by using the *Convolution Theorem* for the Laplace Transform given by (5) we find from the definition (4) that

$$\mathcal{L}\left[\left(\mathcal{H}^{w;1,n;\alpha}_{0+;p,q;\beta}\varphi\right)(x)\right](s) = \mathcal{L}\left[x^{\beta-1}H^{1,n}_{p,q}[wx^{\alpha}]\right](s) \cdot \mathcal{L}[\varphi(x)](s)$$

$$= s^{-\beta} H^{1,n+1}_{p+1,q}\left[ws^{-\alpha} \,\middle|\, \begin{matrix} (1-\beta,\alpha),(a_j,\alpha_j)_{1,p} \\ (0,1),(b_j,\beta_j)_{2,q} \end{matrix}\right]\Phi(s) \tag{6}$$

$$(\Re(s) > 0;\ \ \alpha > 0;\ \ \Re(\beta) + \min_{1\le j\le m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0),$$

where, for convenience,

$$\Phi(s) := \mathcal{L}[\varphi(x)](s) \qquad (\Re(s) > 0).$$

3. Main Result

A general Volterra-type fractional equation associated with an integral operator with the H-function in its kernel is given by

$$\left(\mathcal{H}^{w;1,n;\alpha}_{0+;p,q;\beta}y\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}y(t)dt := g(x) \tag{7}$$

$$(\Re(\beta) > 0;\ \ w \in \mathbb{C}\backslash\{0\};\ \ 0 \leq n \leq p;\ \ \Re(\beta) + \min_{1 \leq j \leq m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0; \Re(\nu) > 0)$$

has the solution

$$y(x) = \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r + l - \beta - \alpha\mu - 1}}{\Gamma(\alpha r + l - \beta - \alpha\mu)} g^{(l)}(t)dt \tag{8}$$

provided that

$g^{(i)}(0) = 0$ for $0 \le i \le l - 1, l$ being a positive integer and $\nu - \beta$ is an integer. Also

$$E_r = (-1)^r (\lambda_\mu^{-r-1}) w^r \det \begin{bmatrix} \lambda_{\mu+1} & \lambda_\mu & \cdots\cdots & 0 & \cdots\cdot & 0 \\ \lambda_{\mu+2} & \lambda_{\mu+1} & \cdots & \cdot\cdot & \cdots\cdot & 0 \\ . & . & & & & \\ . & . & w^{-\alpha}\left(\lambda_{\mu+\frac{\nu-\beta}{\alpha}} + \frac{a}{w^{\frac{\nu-\beta}{\alpha}}}\right) & & & \\ . & . & & & & \\ \lambda_{\mu+r} & \lambda_{\mu+r-1} & \cdots\cdot & \cdots & \cdots & \lambda_{\mu+1} \end{bmatrix} \tag{9}$$

and μ is the least k for which

$$\lambda_k = \frac{\Gamma(\beta+\alpha k)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j k)}{\prod_{j=2}^{q}\Gamma(1-b_j+\beta_j k)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j k)k!}(-1)^k \neq 0$$

Proof. To solve (7) we first take the Laplace transform of its both sides. Using (6), we get

$$s^{-\beta} H_{p+1,q}^{1,n+1}\left[ws^{-\alpha}\left| \begin{matrix} (1-\beta,\alpha),(a_j,\alpha_j)_{1,p} \\ \\ (0,1),(b_j,\beta_j)_{2,q} \end{matrix}\right.\right] Y(s) + aS^{-\nu}Y(s) = G(s) \tag{10}$$

Using (1) for writing H-function in series as

$$H_{p+1,q}^{1,n+1}\left[ws^{-\alpha}\left| \begin{matrix} (1-\beta,\alpha),(a_j,\alpha_j)_{1,p} \\ \\ (0,1),(b_j,\beta_j)_{2,q} \end{matrix}\right.\right] =$$

$$\sum_{k=0}^{\infty} \frac{\Gamma(\beta+\alpha k)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j k)}{\prod_{j=2}^{q}\Gamma(1-b_j+\beta_j k)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j k)} \frac{(-ws^{-\alpha})^k}{k!}$$

$$= \sum_{k=0}^{\infty} \lambda_k (ws^{-\alpha})^k,$$

we find that

$$s^{-\beta} H_{p+1,q}^{1,n+1}\left[ws^{-\alpha}\left| \begin{matrix} (1-\beta,\alpha),(a_j,\alpha_j)_{1,p} \\ \\ (0,1),(b_j,\beta_j)_{2,q} \end{matrix}\right.\right] + as^{-\nu} =$$

$$s^{-\beta}\left[\sum_{A=0}^{\infty} \lambda_A (ws^{-\alpha})^A + as^{-\nu+\beta}\right] \tag{11}$$

Let μ denote the least k for which $\lambda_k \neq 0$ therefore series given by (1) can be reciprocated. Writing

$$\left[s^{-\beta-\alpha\mu}\left[\sum_{k=0}^{\infty} \lambda_{k+\mu}(ws^{-\alpha})^k + as^{-\nu+\beta}\right]\right]^{-1} = s^{\beta+\alpha\mu} \sum_{r=0}^{\infty} E_r s^{-\alpha r}, \tag{12}$$

(where E_r is given by (9)), (10) takes the form

$$Y(s) = s^{\beta+\alpha\mu-l} \sum_{r=0}^{\infty} E_r s^{-\alpha r}\left[s^{(l)}G(s)\right] \tag{13}$$

Again (13) can be written as

$$L(y(x); s) = L\left[\sum_{r=0}^{\infty} E_r \frac{x^{\alpha r+l-\beta-\alpha\mu-1}}{\Gamma(\alpha r+l-\beta-\alpha\mu)}; s\right] L[g^{(l)}(u); s] \tag{14}$$

Now using Laplace convolution theorem in the R.H.S of (14) we get

$$L[y(x); s] = L\left[\int_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r+l-\beta-\alpha\mu-1}}{\Gamma(\alpha r+l-\beta-\alpha\mu)} g^l(t)dt; s\right] \tag{15}$$

Finally on taking the inverse of the Laplace transform on both sides of (15)we arrive at the desired solution (8)

4. Special Cases

(i) Reducing H-function in the R.H.S of (4) to the Fox-Wright function(see, e.g. [4],[6]) and defining the fractional integral operator as

$$\left(\Psi_{a+;q;\beta}^{w;p;\alpha}\varphi\right)(x) := \int_a^x (x-t)^{\beta-1} \; {}_p\Psi_q\left[\begin{matrix}(a_j, \alpha_j)_{1,p}; \\ \\ (b_j, \beta_j)_{1,q};\end{matrix} \quad w(x-t)^{\alpha}\right]\varphi(t)dt \tag{16}$$

$$(\Re(\beta) > 0;\ \ w \in \mathbb{C}\backslash\{0\};\ \ p \leq q + 1),$$

We find that (7) takes the form as

$$\left(\Psi_{0+;q;\beta}^{w;p;\alpha}y\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}y(t)dt = g(x) \tag{17}$$

$$\Re(\beta) > 0; w \in \mathbb{C}\backslash\{0\}; p \leq q + 1; \Re(\nu) > 0$$

whose solution is given by

$$y(x) = \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r+l-\beta-\alpha\mu-1}}{\Gamma(\alpha r+l-\beta-\alpha\mu)} g^{(l)}(t)dt \tag{18}$$

provided that

$g^{(i)} = 0$ for $0 \leq i \leq l - 1$, l being a positive integer and $\nu - \beta$ is an integer,

E_r is given by (9) and μ is the least n for which

$$\lambda_n = \frac{\Gamma(a_1+\alpha_1 n)\ldots\Gamma(a_p+\alpha_p n)\Gamma(\beta-\alpha n)}{\Gamma(b_1+\beta_1 n)\ldots\Gamma(b_q+\beta_q n)n!} \neq 0 \tag{19}$$

(ii) Again reducing H-function in R.H.S of (4) to Mittag-Leffler function [7] and defining the fractional integral operator as follows:

$$\left(\xi_{0+;\alpha,\beta}^{w;\gamma,\kappa} y\right)(x) = \int_0^x (x-t)^{\beta-1} E_{\alpha,\beta}^{\gamma,\kappa}[w(x-t)^{\alpha}]y(t)dt, \tag{20}$$

$$(\gamma, w \in \mathbb{C}; \Re(\alpha) > max\{0, \Re(\kappa)-1\}; min\{\Re(\beta), \Re(\kappa)\} > 0)$$

then (7) can be written as

$$\left(\xi_{0+;\alpha,\beta}^{w;\gamma,\kappa} y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt = g(x) \tag{21}$$

$$(\gamma, w \in \mathbb{C}; \Re(\alpha) > max\{0, \Re(\kappa)-1\}; \quad min\{\Re(\beta), \Re(\kappa)\} > 0; \Re(\nu) > 0)$$

whose solution is given by

$$y(x) = \int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r+l-\beta-\alpha\mu-1}}{\Gamma(\alpha r+l-\beta-\alpha\mu)} g^{(l)}(t)dt \tag{22}$$

provided that

$g^{(i)} = 0$ for $0 \leq i \leq l-1, l$ being a positive integer and $\nu - \beta$ is an integer,

E_r is given by (9) and μ is the least n for which

$$\lambda_n = \frac{\Gamma(\gamma+\kappa n)}{\Gamma(\gamma)n!} \neq 0 \tag{23}$$

(iii) Substituting $g(x) = x^2$ in (3.1) we get

$$\left(\mathcal{H}_{0+;p,q;\beta}^{w;1,n;\alpha} y\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} y(t)dt := x^2 \tag{24}$$

$$(\Re(\beta) > 0; \; w \in \mathbb{C}\backslash\{0\}; \; 0 \le n \le p; \; \Re(\beta) + \min_{1 \le j \le m}\left\{\Re\left(\frac{\alpha b_j}{\beta_j}\right)\right\} > 0; \quad \Re(\nu) > 0)$$

has the solution

$$y(x) = 2\int_0^x \sum_{r=0}^{\infty} E_r \frac{(x-t)^{\alpha r-\beta-\alpha\mu+1}}{\Gamma(\alpha r-\beta-\alpha\mu+2)} dt, \tag{25}$$

where E_r is given by (9)

It is interesting to note that, if we put $a = 0$ and substitute a product of general class of polynomials and the H-function in several variables as kernel of the integral operator in the main result, we get the result obtained by Gupta et al[3]. Also the result obtained by Tomovski et al. ([8], Eqs. (16) and (17)) and convolution integral equation given in ([5], p.33, see also [1]) are special cases of our main result.

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