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ON GENERALIZED ϕ-RECURRENT LORENTZIAN α-SASAKIAN MANIFOLDS

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Abstract : We study generalized ϕ-Recurrent Lorentzian α-Sasakian Manifolds. A relation between associated 1-form A & B and relation between characteristic vector field ξ and the vector field ρ_1 and ρ_2 for a generalized ϕ-Recurrent.

Keywords: Lorentzian α-Sasakian Manifolds, generalized ϕ-Recurrent.

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1. Introduction

In [10], S.Tanno classified connected almost contact metric manifolds whose automorphism group possesses the maximum dimension. For such a manifold, the sectional curvature of a plain sections containing ζ is a constant, say c .He showed that they can be divided into three classes:

(1.1) homogeneous normal contact Riemannian manifolds with $c < 0$,

(1.2) global Riemannian products of a line or a circle with a Kaehlar manifold of constant holomorphic sectional curvature if $c = 0$ and

(1.3) a warped product space $\Re \times_f C$ if $c > 0$.

It is well known that the manifolds of class (1.1) are characterized by admitting a Sasakian structure. Kenmotsu [5] characterized the differential geometric properties of the manifolds of class (1.3); the structure so obtained is now known as Kenmotsu structure. In general these structures are not Sasakian [5].The Gray-Hervella classication of almost Hermitian manifolds [4], there appears a class W_4, of Hermitian manifolds which are closely related to locally conformal Kaehlar manifolds [3]. An almost contact metric structure on the manifold M is called a trans-Sasakian structure [4] if the product manifold $M \times \mathfrak{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ (see [6], [7]) coincides with the class of trans-Sasakian structure of type (α, β). We note that trans-Sasakian structure of type $(0,0), (0,\beta)$ and $(\alpha, 0)$ are cosymplectic [2], β-Kenmotsu [5] and α-Sasakian [5] respectively.

In 2005, Ahmet Yildiz [12] studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Ssaskian manifolds are locally isometric with a sphere.In this paper we shall study some properties of generalized ϕ-Recurrent Lorentzian α-Sasakian manifolds and generalized concircular ϕ-Recurrent Lorentzian α-Sasakian manifolds.

The paper is organised as follows: 2 contains preliminaries of Lorentzian α -Sasakian manifolds. In 3 and 4 we obtain results for generalized ϕ-recurrent and generalized concircular ϕ-recurrent Lorentzian α –Sasakian Manifolds.

2. Preliminaries

A $(2n+1)$ dimensional manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1,1)$ a vector field ξ, a 1-form η and Lorentzian metric g which satisfy the following conditions,

$$(a)\ \phi^2 = -I + \eta \otimes \xi, \quad (b)\ \eta \circ \xi = 1, \quad (c)\ \phi \circ \xi = 0, \quad (d)\ \eta \circ \phi = 0, \tag{1}$$

$$(a)\ g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),\ (b)\ g(X,\xi) = \eta(X), \tag{2}$$

$$(\nabla_X \phi)Y = \alpha[g(X,Y)\xi + \eta(Y)X], \tag{3}$$

for all $X,Y \in \chi(M)$ and for smooth functions α on M, ∇ denotes covariant differentiation operator with respect to Lorentzian metric.

For a Lorentzian α−Sasakian manifold,it can be shown that,

$$\eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{4}$$

$$(\nabla_X\xi) = \alpha\phi X, \tag{5}$$

$$(\nabla_X\eta)Y = \alpha g(X,\phi Y), \tag{6}$$

$$R(X,Y)\xi = \alpha^2(\eta(Y)X - \eta(X)Y), \tag{7}$$

$$R(\xi,X)Y = \alpha^2(g(X,Y)\xi - \eta(Y)X), \tag{8}$$

$$R(\xi,Y)\xi = \alpha^2(\eta(X)\xi - X, \tag{9}$$

$$S(X,\xi) = 2n\alpha^2\eta(X), \tag{10}$$

$$QX = 2n\alpha^2X, \tag{11}$$

$$S(\xi,\xi) = 2n\alpha^2, \tag{12}$$

where R,S and Q represents curvature tensor, Ricci tensor and Ricci map respectively.

3. Generalized ϕ-recurrent Lorentzian α-Sasakian manifolds

Definition 1. *Lorentzian α-sasakian manifolds is called generalized ϕ-recurrent Lorentzian α-sasakian manifolds if its curvature tensor R satisfies the condition*

$$\phi^2\big((\nabla_W R)(X,Y)Z\big) = A(W)R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{13}$$

where A and B are two 1-forms, B is non zero, and these are defined by

$$A(W) = g(W,\rho_1), \qquad B(W) = g(W,\rho_2), \tag{14}$$

and ρ_1, ρ_2 are vector fields associated with 1-form A and B respectibely.

Let us consider a generalized ϕ-recurrent Lorentzian α-sasakian manifolds then by virtue of (1) and (13) we have,

$-(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi = A(W)R(X,Y)Z + B(W)[g(Y,Z)X -$
$g(X,Z)Y]$, (15)

From above it follows that,

$$-g\big((\nabla_W R)(X,Y)Z,U\big)+\ \eta\big((\nabla_W R)(X,Y)Z\big)\eta(U)=\qquad A(W)g(R(X,Y)Z,U)+ B(W)[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)], \tag{16}$$

Let $(e_i), i=1,2,\dots\dots(2n+1)$ be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X=U=e_i$ in (16) and taking summation over i, $1\le i\le 2n+1$, we get

$$-(\nabla_W S)(Y,Z)+\textstyle\sum_{i=1}^{2n+1}\eta\left((\nabla_W R)(e_{i,}Y)Z\right)\eta(e_i)=A(W)S(Y,Z)+2nB(W)g(Y,Z), \tag{17}$$

put $Z=\xi$ in above and by virtue of (10) and the equation $g\left((\nabla_W R)(e_{i,}Y)\xi,\xi\right)=0$, we have

$$(\nabla_W S)(Y,\xi)=-2n\alpha^2A(W)\eta(Y)-2nB(W)\eta(Y), \tag{18}$$

Now we have,

$$(\nabla_W S)(Y,\xi)=\nabla_W S(Y,\xi)-S(\nabla_W Y,\xi)-S(Y,\nabla_W\xi), \tag{19}$$

using (5), (6) and (10) in above relation we have,

$$(\nabla_W S)(Y,\xi)=2n\alpha^3g(W,\phi Y)-\alpha S(Y,\phi W), \tag{20}$$

comparing (18) and (20) we have,

$$2n\alpha^3g(W,\phi Y)-\alpha S(Y,\phi W)=-2n\alpha^2A(W)\eta(Y)-2nB(W)\eta(Y), \tag{21}$$

replacing Y by ξ in above and using (10) we have,

$$A(W)\alpha^2+B(W)=0, \tag{22}$$

Theorem 1. *In a generalized ϕ-recurrent Lorentzian α-sasakian manifolds, the 1-form A and B are related as in (22).*

If $\alpha = 1$, then Lorentzian α-sasakian manifolds reduces to sasakian manifolds, then by (22) $A(W) + B(W) = 0$.

Corollary 1 In a generalized ϕ-recurrent sasakian manifolds, the 1-form A and B are related as $A(W) + B(W) = 0$.

If $\alpha = 0$, then by (22) $B(W) = 0$.

Corollary 2 In cosymplectic manifold, generalized ϕ-recurrent reduces to ϕ-recurrent manifold.

Now from (15) we have,

$$(\nabla_W R)(X,Y)Z = \eta((\nabla_W R)(X,Y)Z)\xi - A(W)R(X,Y)Z - B(W)[g(Y,Z)X - g(X,Z)Y], \quad (23)$$

Now by (23) and Bianchi identity we get,

$$A(W)\eta(R(X,Y)Z) + A(X)\eta(R(Y,W)Z) + A(Y)\eta(R(W,X)Z) + B(W)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] + B(X)[g(W,Z)\eta(Y) - g(Y,Z)\eta(W)] + B(Y)[g(X,Z)\eta(W) - g(W,Z)] = 0, \quad (24)$$

using (4) in (24), also putting $Y = Z = e_i$ and taking summation over $1 \leq i \leq 3$, we get,

$$\alpha^2 A(W)\eta(X) = \alpha^2 A(X)\eta(W),\ B(W)\eta(X) = B(X)\eta(W), \quad (25)$$

Theorem 2. *In a generalized ϕ-recurrent Lorentzian α-sasakian manifolds, the characteristic vector field ξ and the vector field ρ_1 and $\rho_2$associated with the 1-form A, B respectivily are in the same direction and the 1-form A, B are given by (25).*

4.Generalized concircular ϕ-recurrent Lorentzian α-Sasakian manifolds

In this section we study three-dimensional generalized concircular ϕ-recurrent Lorentzian α-sasakian manifolds.

Definition 2. *Lorentzian α-sasakian manifolds (M^3, g) is called generalized concircular ϕ-recurrent if its concircular curvature tensor $\bar{C}$*

$$\bar{C}(X,Y)Z = R(X,Y)Z - \frac{r}{6}[g(Y,Z)X - g(X,Z)Y], \tag{26}$$

Satisfies the condition

$$\phi^2\left(\left(\nabla_W\tilde{C}\right)(X,Y)Z\right) = A(W)\tilde{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{27}$$

where $A(W)$ and $B(W)$ are defined above as in (14) and r is the scalar curvature of the manifold (M^3,g).

Let us consider a generalised concircular ϕ-recurrent Lorentzian α-sasakian manifolds.

Then by (2) we have,

$$-\left(\nabla_W\tilde{C}\right)(X,Y)Z + \eta\left(\left(\nabla_W\tilde{C}\right)(X,Y)Z\right)\xi = A(W)\tilde{C}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{28}$$

From which it follows that,

$$-g\big((\nabla_W\bar{C})(X,Y)Z,U\big) + \eta\big((\nabla_W\bar{C})(X,Y)Z\big)\eta(U) = A(W)g(\bar{C}(X,Y)Z,U) + B(W)[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)]. \tag{29}$$

Let e_1, e_2 and e_3 be a local orthonormal basis of the tangent space at any point of the manifold. Then putting $Y = Z = e_i$ in (29) and taking summation over $i, 1 \le i \le 3$, we get

$$\begin{aligned} -(\nabla_W S)(X,U) + \frac{\nabla_W r}{3}g(X,U) + (\nabla_W S)(X,\xi)\eta(U) - \frac{\nabla_W r}{3}\eta(X)\eta(U) \\ = A(W)\left[S(X,U) - \frac{r}{3}g(X,U)\right] + 2B(W)g(X,U) \end{aligned} \tag{30}$$

putting $X = U = \xi$ in (30) we have,

$$A(W)\left[2n\alpha^2 - \frac{r}{3}\right] + 2B(W) = 0. \tag{31}$$

Theorem 3. *In a three-dimensional generalized concircular ϕ-recurrent Lorentzian α-sasakian manifolds, the 1-form A and B are related as in (31).*

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