

SECOND ORDER PARALLEL TENSORS ON (ϵ)-CONTACT MANIFOLDS

SURENDRA YADAV

Department of Applied Mathematics,
Babasaheb Bhimrao Ambedkar University, Lucknow-226025, (India).
Email: ssyp_p@hotmail.com

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Abstract : Levy (1926) has proved that a second order symmetric parallel non-singular tensor on a space of constant curvature is a constant multiple of the metric tensor. In this paper we have shown that a second order symmetric parallel tensor on a (ϵ)- K -contact manifold is a constant multiple of the associated metric tensor and there is no non-zero parallel 2-form on a (ϵ)-Sasakian manifold.

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1. Preliminaries

A differentiable C^∞ $(2n+1)$ -dimensional manifold M is said to be a (ϵ)-contact Manifold if it carries a global differential 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. On a (ϵ)-contact manifold there exists a unique vector field ξ (called the characteristic vector field) such that [3].

$$\eta(\xi) = a^2, \quad (d\eta)(\xi, X) = 0. \quad (1)$$

for any vector field X on M . By polarization one obtains a Riemannian metric g (called an associated metric) and a $(1,1)$ -tensor ϕ on M such that

$$\begin{aligned} (d\eta)(X, Y) &= g(X, \phi Y), \quad \epsilon g(X, \xi) = -\eta(X), \quad \epsilon = g(\xi, \xi), \\ \phi^2 &= a^2 I + \eta \otimes \xi, \end{aligned} \quad (2)$$

for arbitrary vector fields X and Y on M . It follows that

$$\begin{aligned} \phi \xi &= 0, \quad \eta(\phi X) = 0, \quad \text{rank}(\phi) = 2n, \text{ everywhere on } M \\ g(\phi X, \phi Y) + a^2 g(X, Y) + \epsilon \eta(X)\eta(Y) &= 0, \end{aligned} \quad (3)$$

where $\epsilon = \pm 1$. The tensor field ϕ defines naturally a 2-form Φ on M $\Phi(X, Y) = g(X, \phi Y)$.

A (ϵ) -contact manifold is said to be (ϵ) - K -contact manifold if ξ is killing. It is well known that a (ϵ) -contact manifold is (ϵ) - K -contact iff the sectional curvatures of all plane sections containing ξ are equal to 1. On a (ϵ) - K -contact manifold the following formulas hold [4]

$$\nabla_X \xi = -\epsilon \phi X, \quad (4)$$

$$R(\xi, X)\xi = a^2 X + \eta(X)\xi, \quad (5)$$

where ∇ denotes the Riemannian connection of g and R its curvature tensor of type $(1, 3)$. A (ϵ) -contact manifold whose contact structure is normal, i.e. whose torsion $[\phi, \phi] + 2(d\eta) \otimes \xi$ vanishes identically is called a (ϵ) -Sasakian manifold. A (ϵ) -Sasakian manifold is (ϵ) - K -contact but the converse is not necessarily true. A (ϵ) -Sasakian manifold has stronger curvature property than (5)

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (6)$$

$$R(\xi, X)Y = -\epsilon g(X, Y)\xi + \eta(Y)X. \quad (7)$$

Definition 1. A tensor T of second order is said to be a second order parallel if $\nabla T = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g .

Theorem 1. On a (ϵ) - K -contact manifold a second order symmetric parallel tensor is a constant multiple of the associated metric tensor.

Proof. Let α denote a $(0, 2)$ -symmetric tensor field on a (ϵ) - K -contact manifold M such that $\nabla \alpha = 0$. Then it follows that [1]

$$\alpha(R(W, X)Y, Z) + \alpha(Y, R(W, X)Z) = 0 \quad (8)$$

for arbitrary vector fields W, X, Y, Z on M .

Substitution of $W = Y = Z = \xi$ in (2.1) gives us

$$\alpha(\xi, R(\xi, X)\xi) = 0. \text{ (because } \alpha \text{ is symmetric).}$$

As the manifold is (ϵ) - K -contact, use of (5) in the above equation we get

$$a^2\alpha(X, \xi) - \epsilon g(X, \xi)\alpha(\xi, \xi) = 0. \tag{9}$$

Differentiating (9) covariantly along Y , we get

$$\begin{aligned} a^2[\alpha(\nabla_Y X, \xi) + \alpha(X, \nabla_Y \xi)] - \epsilon[g(\nabla_Y X, \xi) + g(X, \nabla_Y \xi)]\alpha(\xi, \xi) \\ - 2\epsilon g(X, \xi)\alpha(\nabla_Y \xi, \xi) = 0. \end{aligned} \tag{10}$$

Putting $X = \nabla_Y X$ in (2.2), we get

$$a^2\alpha(\nabla_Y X, \xi) - \epsilon g(\nabla_Y X, \xi)\alpha(\xi, \xi) = 0. \tag{11}$$

From (10), (11) and (4), we get

$$a^2\alpha(X, \emptyset Y) - \epsilon g(X, \emptyset Y)\alpha(\xi, \xi) - 2\epsilon g(X, \xi)\alpha(\emptyset Y, \xi) = 0. \tag{12}$$

Replacing X by $\emptyset Y$ in (9) and using (2) gives

$$\alpha(\emptyset Y, \xi) = 0. \tag{13}$$

From (12) and (13) it follows that

$$a^2\alpha(X, \emptyset Y) - \epsilon g(X, \emptyset Y)\alpha(\xi, \xi) = 0. \tag{14}$$

Replacing Y by $\emptyset Y$ in (14) and using (3), (2) and (9) we get

$$a^2\alpha(X, Y) = \epsilon \alpha(\xi, \xi)g(X, Y). \tag{15}$$

Differentiating (9) covariantly along any vector field on M , it can be easily seen that $\alpha(\xi, \xi)$ is constant.

Theorem 2. *On a (ϵ) -Sasakian manifold there is no non-zero parallel 2-form.*

Proof. Let α be a parallel 2-form on a (ϵ) -Sasakian manifold M and let A be a (1,1)-tensor field metrically equivalent to α , i.e. $\alpha(X, Y) = g(AX, Y)$. Then (2.1) holds because (8) is true of any second order parallel tensor. Substituting $X = Y = \xi$ in that gives [1]

$$g(A(R(W, \xi)\xi), Z) + g(A\xi, R(W, \xi)Z) = 0. \tag{16}$$

Using (4), (5) and (7) in (16), we obtain

$$-a^2g(AW, Z) - \eta(W)g(A\xi, Z) + \eta(Z)g(A\xi, W) + g(W, Z)g(A\xi, \xi) = 0 \tag{17}$$

Since α is a 2-form we have

$$g(A\xi, \xi) = 0. \tag{18}$$

Thus from (17) and (18), we have

$$a^2AW = \epsilon [g(W, \xi)A\xi - g(A\xi, W)\xi]. \quad (19)$$

Putting $A\xi$ for ξ in (19), we get

$$AW = 0. \quad (20)$$

From (20), we get $A = 0$. This completes the proof.

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