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A FEEDBACK QUEUEING MODEL WITH IMPATIENT CUSTOMERS

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Abstract : In this paper, we consider a single server feedback queueing model with impatient customers. Inter-arrival time, service time and impatience of customers follow exponential distribution. Steady –state queue length probabilities are obtained. Laplace transform of generating function of transient-state queue length probabilities is also obtained. Some important cases of interest are derived as special cases.

Keywords : Feedback, impatient customers, Laplace transform, generating function, exponential distribution.

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1. Introduction

Queueing models have been effectively used in the design and analysis of telecommunication system, traffic system, service system and many more. A number of extensions in basic queueing models have been made and the concepts like vacation queueing, correlated queueing, queueing with feedback, queueing with impatience and catastrophic queueing have come up. Of these, ‘Feedback’ in queueing literature represents customer’s dissatisfaction because of inappropriate quality of service. In case of Feedback, customers arrive and stand in queue for service. After each service, customers either leave the system or rejoin the queue. Finch [7] introduced the concept of feedback through his paper ‘cyclic queue with feedback’. Delbruck [6] studied an online random linear network under Poisson arrivals, variable bulk service and feedback.Sharda and Garg [7] also obtained the timedependent solution of a queueing model with feedback.

There is now growing interest in the analysis of queueing system with impatient customers. This is due to the vastapplications in business world, communication centers and call centers. This impatience finds reflection in two ways namely balking and reneging. In balking, customers are refusing to join the queue because it has reached a certain length. Balking affects the arrival rate. On the other hand,reneging customers join the queue and waiting for service, if the perceived waiting time exceed the customers expectation then the customer leaves the queue. The reneging customer affects the service rate. Haight [10] studied a queue with reneging in which he studied- a) the problem like how to make rational decisions while waiting in queues, b) the probable effect of this decision and c)behavior of a queue in which all persons are employing such a procedure. Ancker and Gaferian [3] studied queueing model with balking and reneging and obtained steady-state probabilities, mean number in the queue, probabilities of balking, reneging, waiting etc. Choudhury andMedhi [4] analysed M/M/1/K markovian queueing model with balking and a position dependent reneging and steady-state probabilities are derived.

Palm [13], Reynolds [14], Cox [5], Jain and Singh [11] are worked on reneging and gave the stationary solution of a multi-server queueing model with discouragement. Abou-El-Ata and Hariri [1] studied multi-server finite capacity Markovian queue with balking and reneging. Alseedy et al. [2] studied M/M/c queue with balking and reneging and derived its transient solution by using the probability generating function technique and the properties of Bessel function.

Santhakumaranand Thangaraj [15] studied M/M/1feedback queueing models with impatient for queue length at arrival epoch and obtained results for stationary distribution. Kumar and Sharma studied M/M/1 /N feedback queueing modelwith retention of reneged customers and obtained the steady-state solution of the model.

In the present paper, we have extended the work of Garg and Singla [8] for single server by applying the concept of reneging. We consider a single server queueing model with infinite waiting capacitywherein an arrival either leaves the system or re-joins it with constant probability after being served once. The customers however leave the system definitely after having received the service for the second time. Sometimesa customer leaves the queue (renege) due to impatience before getting their service. Thus we consider$M/M/1/\infty$ feedback queueing model with impatient customers. Steady-state queue length probabilities are obtained.Laplace transform of the probability generating function of transient-state queue length probabilities is also obtained.

The practical situation which corresponds to the above model can be that of a service station, whereinthe customers arrived for repair or service of their vehicles.After being served once, some customers are not satisfied due to ill-mannered services and so they rejoined the service station again. Some customers have not enough time for waiting so they reneges the queue. The manager of a service station can know the various probabilities of the number of vehicles to be serviced or repaired by any time.

The queueing system studied in this paper is described by the following assumptions

(i) Arrivals follow the Poisson distribution with parameter λ and the service time distribution of every unit is exponential with parameter μ.

(ii) The probability of rejoining the system is ‘p’ and that of leaving the system is ‘q’ for the units getting first service, so that$p + q = 1$.However the units will have to leave the system after getting second service.

(iii) The unit standing at the head of the line join the service channel for the first time with probability c_1 and for the second time with probability c_2, so that $c_1 + c_2 = 1$.

(iv) The probability of a unit reneging during Δtwhen there are n units in the queue is r(n) Δt and also assumed that reneging follow the exponential distribution. If any one of the (n-1) customers in the queue renege then the density function for the minimum of (n-1) selections from d(t) becomes

$$d_{n-1}(t) = (n-1)\alpha e^{-(n-1)\alpha t}$$

$$\text{Also} \quad r(n) = (n-1)\alpha$$

(v) The waiting space is infinite.

(vi) The stochastic processes involved, viz

(a) Arrival of units (b) Departure of units

are statistically independent.

2. Definitons

$P_n^{(0)}(t)$ = Probability that there are n units in the system at any time t and next unit is to depart for the first time.

$P_n^{(1)}(t)$ = Probability that there are n units in the system at any time t and next unit is to depart for the second time.

$P_n(t)$ =Probability that there are n units in the system at any time t.

$$P_n(t) = P_n^{(0)}(t) + P_n^{(1)}(t) \quad , n \geq 0 \qquad \dots(1)$$

Initially

$$P_0^{(0)}(t) = 1 \quad \text{and} \quad P_0^{(1)}(t) = 0 \quad , t \geq 0 \text{ and } r(1) = 0$$

The difference – differential equations describing the system are

$$\frac{d}{dt}P_0^{(0)}(t) = -\lambda P_0^{(0)}(t) + \mu q P_1^{(0)}(t) + \mu P_1^{(1)}(t) \qquad \dots(2)$$

$$\frac{d}{dt}P_n^{(0)}(t) = -(\lambda + \mu + (n-1)\alpha)P_n^{(0)}(t) + \lambda P_{n-1}^{(0)}(t) + \mu c_1 p\left(1 - \delta_{n,1}\right)P_n^{(0)}(t) + (\mu c_1 q + n\alpha)P_{n+1}^{(0)}(t) + \mu c_1 P_{n+1}^{(1)}(t) , n \geq 1 \qquad \dots(3)$$

$$\frac{d}{dt}P_n^{(1)}(t) = -(\lambda + \mu + (n-1)\alpha)P_n^{(1)}(t) + \lambda P_{n-1}^{(1)}(t) + (\mu c_2 + n\alpha)P_{n+1}^{(1)}(t) + \mu p(c_1\delta_{n,1} + c_2)P_n^{(0)}(t) + \mu c_2 q P_{n+1}^{(0)}(t)\ , n \geq 1 \qquad \ldots(4)$$

where $\delta_{n,1} = \begin{cases} 1, & \text{for } n = 1 \\ 0, & \text{otherwise} \end{cases}$

The steady- state difference equations describing the system are

$$\lambda P_0^{(0)} = \mu q P_1^{(0)} + \mu P_1^{(1)} \qquad \ldots(5)$$

$$(\lambda + \mu + (n-1)\alpha)P_n^{(0)}$$

$$= \lambda P_{n-1}^{(0)} + \mu c_1 p\big(1 - \delta_{n,1}\big)P_n^{(0)} + (\mu c_1 q + n\alpha)P_{n+1}^{(0)} + \mu c_1 P_{n+1}^{(1)}, n \geq 1 \qquad \ldots(6)$$

$$(\lambda + \mu + (n-1)\alpha)P_n^{(1)}$$

$$= \lambda P_{n-1}^{(1)} + (\mu c_2 + n\alpha)P_{n+1}^{(1)} + \mu p\big(c_1\delta_{n,1} + c_2\big)P_n^{(0)} + \mu c_2 q P_{n+1}^{(0)}, n \geq 1 \qquad \ldots(7)$$

3. Steady - state solution of the problem

Using E f(x) = f(x+1), Eqs. (6) and (7) give

$$[(\mu c_1 q + n\alpha)E^2 + \{\mu c_1 p - (\lambda + \mu + (n-1)\alpha)\}E + \lambda]P_n^{(0)} + \mu c_1 E^2 P_n^{(1)} = 0\ , n \geq 2 \qquad \ldots(8)$$

$$[\mu c_2 q E^2 + \mu c_2 p E]P_n^{(0)} + [(\mu c_2 + n\alpha)E^2 - \{\lambda + \mu + (n-1)\alpha\}E + \lambda]P_n^{(1)} = 0\ , n \geq 2 \qquad \ldots(9)$$

Solving (8) and (9) with help of determinants, we have

$$[n\alpha E^2 - \{\lambda + \mu + (n-1)\alpha\}E + \lambda][(\mu c_1 q + \mu c_2 + n\alpha)E^2 - \{\lambda + \mu + (n-1)\alpha - \mu c_1 p\}E + \lambda] = 0, n \geq 2 \qquad \ldots(10)$$

The two roots of (10) are obtained by solving its first factor and we get

$$E = \frac{\lambda + \mu + (n-1)\alpha \pm \sqrt{\{\lambda + \mu + (n-1)\alpha\}^2 - 4n\lambda\alpha}}{2n\alpha}$$

$$= \frac{\lambda + \mu + (n-1)\alpha \pm \sqrt{\{\lambda + \mu - (n-1)\alpha\}^2 + 4\alpha(\mu(n-1) - \lambda)}}{2n\alpha}$$

$$= \frac{\lambda + \mu + (n-1)\alpha \pm \{\lambda + \mu - (n-1)\alpha + \delta\}}{2n\alpha}$$

where δ is a small +ve quantity.

Therefore $z_0 = \frac{2(n-1)\alpha - \delta}{2n\alpha}$ and $z_1 = \frac{2(\lambda+\mu)+\delta}{2n\alpha}$ ...(11)

z_0 is always less than 1whatever may be the value of various parameters but z_1 is less than 1 only when $2(\lambda+\mu) + \delta < 2n\alpha$. The other two roots z_2, z_3 are obtained from

$[(\mu c_1 q + \mu c_2 + n\alpha)E^2 - (\lambda + \mu + (n-1)\alpha - \mu c_1 p)E + \lambda] = 0$

after putting the values of the parameters λ, μ, $c_1, c_2 p$, q, in the quadratic equation. After evaluating z_2, z_3; we find if any of these ≥ 1, then to have convergence of solution that root ≥ 1 must be rejected.

The value of $P_n^{(0)}$ and $P_n^{(1)}$ are given by

$$P_n^{(0)} = \sum_{i=0}^{3} a_i z_i^n \text{ and } P_n^{(1)} = \sum_{i=0}^{3} b_i z_i^n \quad \text{for } n \geq 2$$

where z_0, z_1, z_2, z_3 are the roots of equation (10) and a_i, b_i(i=0, 1, 2, 3) are arbitrary constants to be evaluated. . In case $z_i \geq 1$ take $a_i, b_i = 0$; i = 1, 2, 3.

Now from (5) and {(6) for n = 1}, we find probability $P_1^{(0)}$ and $P_1^{(1)}$ in terms of $P_0^{(0)}$:

$$P_1^{(0)} = \frac{1}{\lambda + \mu}\left[(\mu c_1 q + \alpha)\sum_{i=0}^{3} a_i z_i^2 + \mu c_1 \sum_{i=0}^{3} b_i z_i^2 + \lambda P_0^{(0)}\right] \quad ...(12)$$

$$P_1^{(1)} = \frac{1}{\lambda+\mu}\left[\lambda(\lambda+\mu p)P_0^{(0)} - \mu q[\mu c_1 q+\alpha)\sum_{i=0}^{3} a_i z_i^2 + \mu c_1 \sum_{i=0}^{3} b_i z_i^2\right] \qquad \ldots(13)$$

Eight unknown $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ (two unknown a_0, b_0 in case $z_i>1$; i = 1,2,3) can be evaluated from equation (6) for n=2, 3,4,5 and (7) for n=1,2,3,4 in terms of $P_0^{(0)}$ and the value of $P_0^{(0)}$ can be found by using the relation

$$P_0^{(0)} = 1 - \sum_{n=1}^{\infty} (P_n^{(0)} + P_n^{(1)})$$

Hence by using the value of $a_0, a_1, a_2, a_3, b_0 b_1, b_2, b_3$ and $P_0^{(0)}$, the probabilities $P_n^{(0)}$ and $P_n^{(1)}$ are completely known for various value of n.

4. Special cases

(i) When there is no feedback

Putting q = 1; p = 0; $c_1 = 1$; $c_2 = 0$; $P_n^{(1)} = 0$; $P_n^{(0)} = P_n$ then equations (5) – (7) reduce to

$$\lambda P_0 = \mu P_1 \text{ and } (\lambda + \mu + (n-1)\alpha)P_n = \lambda P_{n-1} + (\mu + n\alpha)P_{n+1}$$

Solving the above equation recursively and we get

$$P_n = P_0 \prod_{i=1}^{n} \frac{\lambda}{\mu+(i-1)\alpha}, n \geq 1 \qquad \ldots(14)$$

and the value of P_0 can be found by using the relation

$$\sum_{n=0}^{\infty} P_n = 1$$

Therefore

$$P_0 = \left[1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \frac{\lambda}{\mu+(i-1)\alpha}\right]^{-1} \text{ and}$$

$$P_n = \prod_{i=1}^{n} \frac{\lambda}{\mu + (i-1)\alpha}\left[1 + \sum_{n=1}^{\infty}\prod_{i=1}^{n}\frac{\lambda}{\mu + (i-1)\alpha}\right]^{-1}, (n \geq 1)$$

which coincides with M/M/1 with Reneging given by Gross and Harris [8] for

$r(i) = (i-1)\alpha$ and $b_{i-1} = 1$

(ii) When there is no feedback and no reneging

Putting $\alpha=0$ in equation (14) we get

$$P_n = P_0\left(\frac{\lambda}{\mu}\right)^n, (n \geq 1)$$

This coincides with M/M/1 classical model.

5. Transient- state solution

Let reneging follow exponential distribution with density function $d(t) = \alpha e^{-\alpha t}$

where it is assumed that customer can renege at any time independent of number of units in the queue i.e. $r(n) = \alpha, n > 1$

Taking Laplace transform of equations (2) to (4) and using

$\bar{P}_n(s) = \int_0^{\infty} e^{-st} P_n(t)dt$; Re s > 0, we get

$$(s + \lambda)\bar{P}_0^{(0)}(s) = 1 + \mu q\bar{P}_1^{(0)}(s) + \mu\bar{P}_1^{(1)}(s) \qquad \ldots(15)$$

$$\left(s + \lambda + \mu + \alpha(1 - \delta_{n,1})\right)\bar{P}_n^{(0)}(s) = \lambda\bar{P}_{n-1}^{(0)}(s) + \mu c_1 p(1 - \delta_{n,1})\bar{P}_n^{(0)}(s) + (\mu c_1 q + \alpha)\bar{P}_{n+1}^{(0)}(s) + \mu c_1\bar{P}_{n+1}^{(1)}(s), n \geq 1 \qquad \ldots(16)$$

$$\left(s + \lambda + \mu + \alpha(1 - \delta_{n,1})\right)\bar{P}_n^{(1)}(s) = \lambda\bar{P}_{n-1}^{(1)}(s) + (\mu c_2 + \alpha)\bar{P}_{n+1}^{(1)}(s)$$

$$+\mu p\left(c_1\delta_{n,1} + c_2\right)\overline{P}_n^{(0)}(s) + \mu c_2 q\overline{P}_{n+1}^{(0)}(s)\ ,n \geq 1 \qquad \ldots(17)$$

It we define

$$P^{(0)}(z.t) = \sum_{n=0}^{\infty} P_n^{(0)}(t)z^n, \qquad \overline{P}(0)(z,s) = \int_0^{\infty} e^{-st}\, P^{(0)}(z,t)dt$$

$$P^{(1)}(z,t) = \sum_{n=0}^{\infty} P_n^{(1)}(t)z^n\ ,\overline{P}(1)(z,s) = \int_0^{\infty} e^{-st}\, P^{(1)}(z,t)dt$$

$$P(z,t)\ =\ P^{(0)}(z,t) + P^{(1)}(z,t) \text{ and } \overline{P}(z,s) = \int_0^{\infty} e^{-st}P(z,t)dt$$

with $|z| \leq 1$, then Laplace transformation of probability generating function of transient – state queue length probabilities are

$$\overline{P}^{(0)}(z,s) = \frac{1}{B(z)}\Big[\{-(A(z)-\alpha-\mu c_2)(\alpha(1-z)-\mu z)-(A(z)-\alpha)\mu c_1(q+pz)\}\overline{P}_0^{(0)}(s) +$$

$$\{(A(z)-\mu-\alpha)(\mu c_2 qz-\mu c_1 pz^2-\alpha z(1-z))-\mu c_1\alpha z(1-z)\}\overline{P}_1^{0}(s)$$

$$+\{\mu c_2 z(A(z)-\alpha-\mu)-\mu c_1\alpha z(1-z)\}\overline{P}_1^{(1)}(s)+z(A(z)-\mu c_2-\alpha)\Big] \quad \ldots(18)$$

$$\overline{P}^{(1)}(z,s) = \frac{1}{B(z)}\Big[-\mu c_2(q+pz)(A(z)-\alpha z-\mu z)\overline{P}_0^{(0)}(s) +$$

$$+\{(\mu c_2 qz-\mu c_1 pz^2)(-A(z)+\alpha+\mu(q+pz)-\mu c_2(p+qz)\alpha z(1-z))\}\overline{P}_1^{0}(s)$$

$$+\{-(A(z)-\alpha)(\mu c_2 z+\alpha z(1-z)+\mu(p+qs)(\mu c_2 z+c_1\alpha z(1-z)\}\overline{P}_1^{(1)}(s)$$

$$+z\mu c_2(q+pz)\Big] \qquad \ldots..(19)$$

$$\overline{P}(z,s) = \frac{1}{B(z)}\Big[-(1-z)[\{-A(z)(\mu q+\alpha)+\alpha(\mu+\alpha)-\mu c_1 p\alpha-\mu c_2 pz(\alpha+\mu)\}\overline{P}_0^{(0)}(s)$$

$$+z\{-A(z)\alpha+\alpha^2-\mu^2p(c_2q-c_1pz)+\mu c_2p\alpha(1-z)\}\overline{P}_1^0(s)$$

$$+z\{-(A(z)\alpha+\alpha^2-\mu^2c_2p-\mu c_1p\alpha(1-z)\}\overline{P}_1^{(1)}(s)]+z\{A(z)-\alpha$$

$$-\mu c_2(1-z)\}] \qquad \text{...(20)}$$

where $A(z) = -\lambda z^2+(s+\lambda+\mu+\alpha)z$

and $B(z)=[A(z)-\mu c_1(q+pz)-\alpha][A(z)-\mu c_2-\alpha]-\mu^2c_1c_2(q+pz)$

Let $D(z) = K_1(z)K_2(z)-\mu^2c_1c_2(q+pz)$

where
$$K_1(z) = (-\lambda z^2+(s+\lambda+\mu+\alpha-\mu c_1p)z-(\mu c_1q+\alpha)$$

$$K_2(z) = (-\lambda z^2+(s+\lambda+\mu+\alpha)z-(\mu c_2+\alpha)$$

Obviously $K_1(z)$ and $K_2(z)$ have two zeros inside the unit circle.

Let $f(z) = K_1(z)K_2(z)$ and $g(z) = \mu^2c_1c_2(q+pz)$

$$|f(z)| = |K_1(z)\,K_2(z)|$$

$$= |(-\lambda z^2+(s+\lambda+\mu+\alpha-\mu c_1p)z-(\mu c_1q+\alpha)||(-\lambda z^2+(s+\lambda+\mu+\alpha)z-(\mu c_2+\alpha)|$$

$$\geq (\xi+\mu c_2)(\xi+\mu c_2) \quad \text{for} \quad s=\xi+i\eta,\ |z|=1$$

$$\geq \mu^2c_1c_2 \geq |g(z)|$$

Hence $|f(z)| \geq |g(z)|$ on $|z| = 1$

Since all the condition of Rouche's theorem are satisfied, so D has two zeroes inside the unit circle. Let these zeroes be $z_m(m = 0,1)$. Numerator must also vanish for these two zeroes since $\overline{P}(z,s)$ is analytical function of z. These two equations along with equation

(15) will determine the three unknown's$\overline{P}_0^{(0)}(s), \overline{P}_1^{(0)}(s), \overline{P}_1^{(1)}(s)$ .Hence the generating function $\overline{P}(z,s)$ is completely known.

$\overline{P}_n(s)$ can be obtained by using the following formula

$$\overline{P}_n(s) \quad = \frac{1}{n!}\frac{d^{(n)}\overline{P}(z,s)}{dz^n} \qquad \text{at } z = 0$$

In either case $P_n(t)$can be found by inverting the Laplace transform $\overline{P}_n(s)$. Further

$\overline{P}(1,s) = \frac{1}{s}$, as desired and $\overline{P}(0,s) = \overline{P}_0^{(0)}(s)$.

6. Special cases

(i) When there is no feedback

Putting q = 1; p =0; $\overline{P}^{(1)}(z,s) = 0$; $\overline{P}^{(0)}(z,s) = \overline{P}(z,s)$; $\overline{P}_0^{(0)}(s) = \overline{P}_0(s)$;$\overline{P}_n^{(0)}(s) = \overline{P}_n(s)$; $\overline{P}_n^{(1)}(s) = 0$ in (20) we get

$$\overline{P}(z,s) \;=\; \frac{z-(1-z)[(\mu+\alpha)\overline{P}_0(s)+\alpha z\,\overline{P}_1(s)}{-\lambda z^2+(s+\lambda)z+(\mu+\alpha)(z-1)} \qquad \ldots(21)$$

(ii) When there is no feedback and no reneging.

Putting α = 0 in (21), we get

$$\overline{P}(z,s) \;=\; \frac{z-(1-z)\mu\overline{P}_0(s)}{-\lambda z^2+(s+\lambda+\mu)z-\mu} \qquad \ldots(22)$$

which is same as M/M/1 for transient state.

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