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LEGENDRE CURVE ON THREE-DIMENSIONAL QUASI SASKIAN MANIFOLDS

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Abstract : The object of the present paper is to study locally φ-symmetric Legendre curves and bi-harmonic Legendre curves on three-dimensional quasi-Sasakian manifolds.

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1. Introduction

The notion of quasi-Sasakian manifold was introduced by Blair [2]. Again .Olszak studied quasi-Sasakian manifolds and characterized conformally flat quasi-Sasakian manifolds (see e. g.[7],[8]) The study of Legendre curves was introduced by .Baikoussis and Blair [1]. Again the Legendre curves have been studied on almost contact manifolds by Welyczko [10]. Bi-harmonic curves on contact manifolds have been studied by Cho et al [4]. Further references can be found there. The present author has introduced the notion of locally φ symmetric Legendre curves in the paper [9].

The present paper is organised as follows :

After the introduction in §1, we give some preliminaries in §2. In Section §3 ,we study locally φ-symmetric Legendre curves on three-dimensional quasi-Sasakian manifolds. In §4, we consider bi-harmonic Legendre curves on three-dimensional quasi-Sasakian manifolds. The last section contains an example.

2. Preliminaries

Let M be a (2n+1) dimensional differentiable manifold with an almost contact structure (ϕ,ξ,η,g), where φ is a tensor field of type (1,1), ξ is a unit vector field, η is an 1-form and g is the Riemannian metric, then [3] the following holds.

$$\phi^2(X) = -X + \eta(X)\xi,\ \eta(\xi) = 1, \qquad \ldots(1)$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad \ldots(2)$$

$$\phi(\xi) = 0,\ \eta(\phi X) = 0,\ \eta(X) = g(X,\xi) \qquad \ldots(3)$$

where X,Y are any vector fields on M.

Let Φ be the fundamental 2-form on M defined by $\Phi(X,Y)=g(X,\phi Y)$. M is said to be quasi-Sasakian if it is normal and ϕ is closed. The above facts are also true for three-dimensional quasi-Sasakian manifolds. In this paper we are interested in three-dimensional quasi-Sasakian manifolds. For a three-dimensional quasi-Sasakian manifold we have [6].

$$\nabla_X \xi = -\beta\phi X, \qquad (4)$$

$$\begin{aligned}R(X,Y)Z &= g(Y,Z)[(\tfrac{r}{2} - \beta^2)X + (3\beta^2 - \tfrac{r}{2})\eta(X)\xi + \eta(X)(\phi\,\mathrm{grad}\beta) - d\beta(\phi X)\xi] \\ &- g(X,Z)[(\tfrac{r}{2} - \beta^2)Y + (3\beta^2 - \tfrac{r}{2})\eta(Y)\xi + \eta(Y)(\phi\,\mathrm{grad}\beta) - d\beta(\phi Y)\xi] \\ &+ [(\tfrac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \tfrac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) - \eta(Z)d\beta(\phi Y)]X \\ &- [(\tfrac{r}{2} - \beta^2)g(X,Z) + (3\beta^2 - \tfrac{r}{2})\eta(X)\eta(Z) - \eta(X)d\beta(\phi Z) - \eta(Z)d\beta(\phi X)]Y \\ &- \tfrac{r}{2}[g(Y,Z)X - g(X,Z)Y], \qquad \ldots(5)\end{aligned}$$

where R is the Riemannian curvature tensor of the manifold, r is the scalar curvature of the manifold and β is a function defined on the manifold.

A curve γ is called a Legendre curve if $\eta(\dot\gamma)=0$. For a Legendre curve γ we have from Frenet-Serret formulae

$$\nabla_T T = \kappa N, \qquad \ldots(6)$$

$$\nabla_T N = -\kappa T + \tau B, \qquad \ldots(7)$$

$$\nabla_T B = -\tau N \qquad \ldots(8)$$

where $T=\dot{\gamma}$, (T,N,B) is a Frenet frame in three-dimension, κ=curvature of the curve, τ=torsion of the curve. The curve is a circle if κ=a positive constant and $\tau=0$.

3. Locally ϕ symmetric Legendre curves on three-dimensional quasi-Sasakian manifolds

In this section we like to study locally ϕ symmetric Legendre curves on three-dimensional quasi-Sasakian manifolds. The concept of locally □-symmetric Legendre curves has been introduced in the paper [9].

Definition 3.1 : *A Legendre curve γ on a three-dimensional quasi-Sasakian manifold is called locally ϕ symmetric if it satisfies*

$$\phi^2(\nabla_T R)(\nabla_T T,T)T = 0, \qquad \ldots(9)$$

where $T = \dot{\gamma}$.

Now, $(\nabla_T R)(\nabla_T T,T)T = \kappa \nabla_T R(N,T)T - \kappa\tau R(B,T)T - \kappa^2 R(N,T)N$

Using (5), we get from above

$$(\nabla_T R)(\nabla_T T,T)T = \kappa \nabla_T R(N,T)T - \kappa\tau R(B,T)T - \kappa^2 R(N,T)N$$

$$= \kappa\{d(\tfrac{r}{2} - \beta^2)(T)N + (\tfrac{r}{2} - \beta^2)\nabla_T N - d(d\beta(\phi N))(T)B - d\beta(\text{□}N)\nabla_T B + d(\tfrac{r}{2} - \beta^2)(T)N + (\tfrac{r}{2} - \beta^2)\nabla_T N - d(\tfrac{r}{2})(T)N - \tfrac{r}{2}\nabla_T N\} - \kappa\tau\{(\tfrac{r}{2} - \beta^2)B + (3\beta^2 - \tfrac{r}{2})B + (\phi\,\mathrm{grad}\beta - d\beta(\phi B)B) + (\tfrac{r}{2} - \beta^2)B + d\beta(\phi N)T - \tfrac{r}{2}B\} + \kappa^2\{(\tfrac{r}{2} - \beta^2)T - \tfrac{r}{2}T\}$$

Hence, after a straight forward calculation, we get

$$\phi^2(\nabla_T R)(\nabla_T T,T)T = -\kappa\{2d(\tfrac{r}{2} - \beta^2)(T)N + 2(\tfrac{r}{2} - \beta^2)(-\kappa)T + d\beta(\phi T)\tau N - d(\tfrac{r}{2})(T)N - \tfrac{r}{2}\kappa T\} - \kappa^2\beta^2 T$$

Thus $\phi^2(\nabla_T R)(\nabla_T T,T)T = 0$ implies

$$-\kappa\{2d(\tfrac{r}{2} - \beta^2)(T)N - 2\kappa(\tfrac{r}{2} - \beta^2)T + \tau d\beta(\phi N)N - d(\tfrac{r}{2})(T)N + \tfrac{r}{2}\kappa T\} - \kappa^2\beta^2 T = 0. \qquad \ldots(10)$$

Taking inner product with T, we get from (10).

$$-2\kappa(\frac{r}{2} - \beta^2) + \frac{r}{2}\kappa - \kappa^2\beta^2 = 0.$$

If $\kappa \neq 0$, the above equation yields

$$\kappa = 2 - \frac{r}{2\beta^2}.$$

Thus we are in the position to state the following

Theorem 3.1. *The curvature of a non-geodesic locally* ϕ *symmetricLegendrecurveon a 3-dimensional quasi-Sasakian manifold is given by*

$$\kappa = 2 - \frac{r}{2\beta^2}.$$

If $\beta=0$, the manifold is called cosymplectic manifold. Then from above, we obtain the following

Corollary 3.1.*There exists no non-geodesic locally □-symmetric Legendre curve on a 3-dimensional cosymplectic manifold.*

4. Biharmonic Legendre curves on three-dimensional quasi-Sasakian manifolds

In this section we like to study biharmonic Legendre curves on three-dimensional quasi-Sasakian manifolds. For the definition ofbiharmonic Legendre curves, we have followed the paper [5].

Definition 4.1 :*A Legendre curve* γ *on a three-dimensional quasi-Sasakian manifold is called biharmonicif it satisfies*

$$\nabla_T^3 T + R(\nabla_T T, T)T = 0 \qquad \ldots(11)$$

From (5) and Serret-Frennet formulae, we get

$$R(\nabla_T T, T)T = \kappa\{(\frac{r}{2} - 2\beta^2)N - d\beta(\phi N)B\}.$$

Again, from Serret-Frennet formulae, we get

$$\nabla_T^3 T = -3\kappa\kappa' T + (\kappa'' - \kappa^3 - \kappa\tau^2)N + 2(\tau\kappa' + \kappa\tau')B.$$

Using the above equations in (11), we get

$$-3\kappa\kappa' T + \{(\kappa'' - \kappa^3 - \kappa\tau^2) + \kappa(\frac{r}{2} - 2\beta^2)\}N + \{2(\tau\kappa' + \kappa\tau') - \kappa\phi N(\beta)\}B = 0.$$

Taking inner product with T, we get from above equation

$$-3\kappa\kappa' = 0.$$

Suppose, the curve is non-geodesic, i.e. $\kappa \neq 0$. Hence, it follows that κ = a non-zero constant. Again, taking inner product with N, we get

$$\kappa^2 + \tau^2 = 2\beta^2 - \frac{r}{2}.$$

Thus, we are in a position to state the following

Theorem 4.1. *A non-geodesic biharmonic Legendre curve on a three-dimensional quasi-Sasakian manifold is a circle satisfying*

$$\kappa^2 + \tau^2 = 2\beta^2 - \frac{r}{2}.$$

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