

TRANSIENT STRESS WAVE PROPAGATION IN NON-HOMOGENEOUS SEVEN-PARAMETER VISCOELASTIC FILAMENT

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Abstract : In this paper, eikonal equations of geometric optics and asymptotic methods have been used to find the equation of wave front of transient stress wave in non-homogeneous thin seven-parameter viscoelastic filament. Here the mechanical properties and density are taken as space dependent which vary exponentially. Special case by taking density, rigidity and viscosity of the filament as a function of cosine is discussed and equation of wave front is thus obtained. The progress of transient stress wave is shown graphically using MATLAB.

Keywords: Transient stress wave, non-homogeneous, varying density, eikonal equation, thin seven-parameter, viscoelastic filament.

2010 Mathematical Subject Classification: 74C10; 74C15; 74C20; 74E05; 74E20

1 Introduction

The study of elastic waves in a non-homogeneous layered medium has been given by Ewing, Jardetzky and Press [5]. It is worthwhile to mention about some problems of linear elastic wave propagation or vibrations where variations of characteristics (elastic coefficients) and geometry (area of cross section of a rod) have been taken in various forms e.g. exponential variation by Rostovtsev and Kharanevskafa

[21], Rao and Goda [20], binomial variation by Singh and Dhaliwal [22], Watanabe [26], integral power law variation by Mukhopadyaya [18], Chandra [3], Narain [19] and Erguven [4]. However in some cases the ratio of the elastic coefficients, poisson ratio or the ratio of density to elastic coefficients is taken as constant. The problem of wave propagation in elastic rods with variable area of cross section have been studied by Tsui [24], Bert and Egle [2] and Handelman and Rubinfeld [7] while Vasudeva and Bhaskara [25] examined the same problems for inhomogeneous elastic rods with varying area of cross sections when variation is taken in the form of a power law along the length of the rod. In general these problems have been treated by the method of separation of variables and integral transformation techniques so that differential equations reduce to some standard differential equations whose solutions are known.

In most of the literature, the problems of non-homogeneity are taken as independent of space coordinate. Moodie [17] has found the asymptotic solutions for cylindrical shear waves in non-homogeneous four parameter viscoelastic media. Singh and Singh [23] used asymptotic method to study the wave propagation in non-homogeneous thin five parameter viscoelastic rod. Acharya et al. [1] has studied vibration of an infinite inhomogeneous transversely isotropic viscoelastic medium with a cylindrical hole.

Recently Kakar et al. [8-11, 13] analyzed four parameter and five parameter viscoelastic rod and filament respectively for different waves and also studied these models under dynamic loadings.

In this paper, we consider the wave propagation in non-homogeneous media, when density ' ρ ', rigidity ' G ', and viscosity ' η ' of the material vary exponentially and have same ratio

$$\frac{\rho}{\rho_0} = \frac{G_1}{G_{01}} = \frac{G_2}{G_{02}} = \frac{\eta_2}{\eta_{02}} = \frac{\eta_3}{\eta_{03}} = \frac{\eta_{3'}}{\eta_{03'}} = \frac{\eta_4}{\eta_{04}} = e^{ax}$$

and secondly

when $\rho = \rho_0 \cos \alpha_1 x, G_1 = G_{01} \cos \alpha_2 x, G_2 = G_{02} \cos \alpha_3 x, G_3 = G_{03} \cos \alpha_4 x,$

$\eta_2 = \eta_{02} \cos \alpha_5 x, \eta_3 = \eta_{03} \cos \alpha_6 x, \eta_{3'} = \eta_{03} \cos \alpha_7 x$ and $\eta_4 = \eta_{08} \cos \alpha_8 x$ space

dependent such that the wave velocity is also space dependent. The problem is solved with eikonal equation when the wave equation is approximated using asymptotic theory. The displacements assumed in the problem are so small that under isothermal conditions, the linear constitutive laws hold. The displacement and stress expressions are solved for time dependent displacement and stress boundary conditions. The filaments are supposed to be initially unstressed and at rest.

2. Formulation of the problem and solution

We create the problem of one-dimensional wave in semi-infinite filament with one end at $x = 0$. The coordinate x is taken to be positive in the direction of the axis of filament. Time is denoted by ' t '. We choose a seven parameter model which when subjected to shearing stress, exhibits elastic, viscous and retarded elastic response. This model is the base for the behavior of our viscoelastic material. The model consists of the series combination of a free spring, Kelvin unit, a three parameter unit (a parallel combination of a dash-pot with Maxwell unit) and a free dash-pot as shown in Fig. 1.

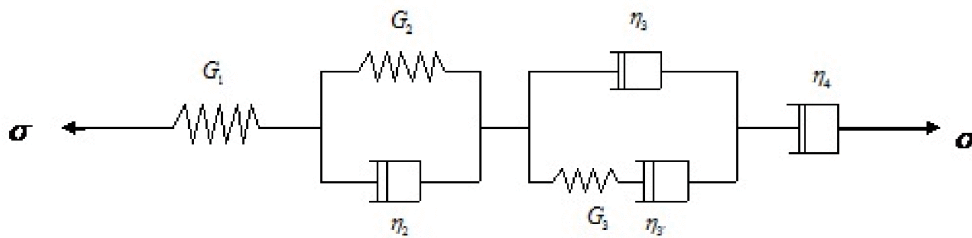


Fig.1 Seven parameter Viscoelastic model.

Let σ be the stress throughout the model and γ be the total strain in the model.

$\gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the strain in section -1, section-2, section-3 and section-4 respectively.

$$\text{So } \gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 \quad \dots (1)$$

The composition equations for stress and strain are as

$$\left. \begin{aligned} \sigma &= G_1 \gamma_1 \\ \sigma &= G_2 \gamma_2 + \eta_2 \frac{\partial(\gamma_2)}{\partial t} \\ \frac{\partial \sigma}{\partial t} + \frac{G_2}{\eta_2} \sigma &= \eta_3 \frac{\partial^2(\gamma_3)}{\partial t^2} + \left(\frac{\eta_3}{\eta_3'} \right) G_3 \frac{\partial(\gamma_3)}{\partial t} \\ \sigma &= \eta_4 \frac{\partial(\gamma_4)}{\partial t} \end{aligned} \right\} \quad \dots (2)$$

Eliminating $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ from (1) and (2), we get the constitutive equation for the model as

$$\left(A_3 \frac{\partial^3}{\partial t^3} + A_2 \frac{\partial^2}{\partial t^2} + A_1 \frac{\partial}{\partial t} + A_0 \right) \sigma = \left(B_3 \frac{\partial^3}{\partial t^3} + B_2 \frac{\partial^2}{\partial t^2} + B_1 \frac{\partial}{\partial t} \right) \gamma \quad \dots (3)$$

where

$$\begin{aligned} A_0 &= \frac{G_2}{\eta_2} \left(\frac{G_2}{\eta_2 \eta_3} + \frac{K}{\eta_4} \right), \quad A_1 = \left(\frac{G_2}{G_1 \eta_2} + \frac{1}{\eta_4} \right) K + \frac{\eta_3}{\eta_2} K + \frac{G_2}{\eta_2} \left(\frac{2}{\eta_3} + \frac{1}{\eta_4} \right), \\ A_2 &= \frac{1}{\eta_2} \left(\frac{G_2}{G_1} + 1 \right) + \frac{K}{G_1} + \left(\frac{1}{\eta_3} + \frac{1}{\eta_4} \right), \quad A_3 = \frac{1}{G_1}, \quad B_1 = \left(\frac{G_1}{\eta_2} K \right), \quad B_2 = \left(\frac{G_1}{\eta_2} + K \right), \\ B_3 &= 1 \text{ and } K = G_3 \left(\frac{1}{\eta_3} + \frac{1}{\eta_3'} \right). \end{aligned} \quad \dots (4)$$

The equation of motion is

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial t^2} \quad \dots (5)$$

The strain-displacement relation is

$$a = \frac{\partial u}{\partial x} \quad \dots (6)$$

where $\rho = \rho(x)$ is the variable density of the material.

Using (5) and (6) in (8), we get

$$\left(A_3 \frac{\partial^4}{\partial t^4} + A_2 \frac{\partial^3}{\partial t^3} + A_1 \frac{\partial^2}{\partial t^2} + A_0 \frac{\partial}{\partial t} \right) \sigma = \begin{pmatrix} \frac{1}{\rho} B_3 \frac{\partial^4 \sigma}{\partial t^2 \partial x^2} - B_3 \frac{\rho'}{\rho^2} \frac{\partial^3 \sigma}{\partial t^2 \partial x} + \frac{B_2}{\rho} \frac{\partial^3 \sigma}{\partial t \partial x^2} - \\ B_2 \frac{\rho'}{\rho} \frac{\partial^2 \sigma}{\partial t \partial x} + \frac{B_1}{\rho} \frac{\partial^2 \sigma}{\partial x^2} - B_1 \frac{\rho'}{\rho^2} \frac{\partial \sigma}{\partial x} \end{pmatrix} \dots (7)$$

The solution of (7) gives the stress field $\sigma(x, t)$ under the approved boundary conditions, where $\sigma(x, t)$ may be represented by the series given by [6] i.e.

$$\sigma(t) = \sum_{n=0}^{\infty} Y_n(x) F_n \{t - h(x)\}, Y_0 \neq 0 \quad \dots (8)$$

where $Y_n(x)$ is the amplitude function, $h(x)$ is the phase function which satisfies the eikonal equation of geometric optics given by [13]

$$\left(\frac{dh(x)}{dx} \right)^2 = \frac{\rho}{G_1} = \frac{1}{c^2} \quad \dots (9)$$

where $c = c(x)$ is the variable wave speed for elastic longitudinal waves in a medium whose modulus is G_1 and F_n 's are given by

$$F_n' = F_{n-1} \quad , \quad n = 1, 2, 3 \quad \dots (10)$$

where prime denotes the differentiation w.r.t whole of the argument $\{t - h(x)\}$. By its successive integrations we can relate all the F_n 's to F_0 .

To find $Y_n(x)$, differentiating (8) and making use of (10), we obtain

$$Y'_n + \frac{1}{2h'} \left\{ \rho A_2 - \frac{\rho'}{\rho} h' - B_2 (h')^2 + h'' \right\} Y_n = Q_n \quad \dots (11)$$

where,

$$Q_n = \frac{1}{2h'} \left[\begin{array}{l} Y''_{n-1} - \left(\frac{\rho'}{\rho} + 2B_2 h' \right) Y'_{n-1} - \left\{ A_1 \rho + B_2 h'' - B_2 \rho' h' - B_1 (h')^2 \right\} Y_{n-1} \\ - B_2 \rho' Y'_{n-2} + B_2 Y'_{n-2} - \left(A_0 \rho + B_1 h'' - B_1 \frac{\rho'}{\rho} h' \right) Y_{n-2} + B_1 Y''_{n-3} - B_1 \frac{\rho'}{\rho} Y'_{n-3} \end{array} \right] \dots (12)$$

By substituting h' from (9) into (11) we get

$$Y'_n + \frac{1}{2} \left[\rho c A_2 - \frac{\rho'}{\rho} - B_2 h' + \frac{h''}{h'} \right] Y_n = Q_n \quad \dots (13)$$

The density and various material characteristics are taken as space-dependent for the case of non-homogeneity of the medium as follows:

$$\frac{\rho}{\rho_0} = \frac{G_1}{G_{01}} = \frac{G_2}{G_{02}} = \frac{\eta_2}{\eta_{02}} = \frac{\eta_3}{\eta_{03}} = \frac{\eta_{3'}}{\eta_{03'}} = \frac{\eta_4}{\eta_{04}} = e^{\alpha x} \quad \dots (14)$$

For $\alpha > 0$ responds to the increasing behavior of characteristics with space co-ordinate, as density, rigidity and viscosity are space dependent but they bear the same ratio. So, it is the case of semi-homogeneous. Using these values in (9), we get

$$\left(\frac{dh(x)}{dx} \right)^2 = \frac{\rho_0}{G_{01}} = \frac{1}{c_0^2} = \text{constant} \quad \dots (15)$$

Integrating the (15) along x -axis, we get

$$h(x) = \frac{x}{c_0} \pm h(0) \quad \dots (16)$$

The plus sign in (16) indicates that the wave progresses in positive direction of x and the negative sign indicates that the wave travels in negative x direction.

Making use of (14) and (15) in (11), we get

$$Y'_n + P_0 Y_n = Q'_n \quad \dots (17)$$

where,

$$P_0 = \rho_0 c_0 A_{02} - \alpha - B_{02}/c_0 = \text{constant}$$

and

$$A_{02} = \frac{1}{\eta_{02}} \left(1 + \frac{G_{02}}{G_{01}} \right) + \frac{G_{03}}{G_{01}} \left(\frac{1}{\eta_{03}} + \frac{1}{\eta_{03'}} \right) + \left(\frac{1}{\eta_{03}} + \frac{1}{\eta_{04}} \right), B_{02} = \frac{G_{01}}{\eta_{02}} + G_{03} \left(\frac{1}{\eta_{03}} + \frac{1}{\eta_{03'}} \right),$$

$$Q'_n(x) = \frac{c_0}{2} \left[Y''_{n-1} - \left(\alpha + 2 \frac{B_{02}}{c_0} \right) Y'_{n-1} - \left\{ A_{01} \rho_0 - \alpha \rho \frac{B_{02}}{c_0} - B_{01} \left(\frac{1}{c_0} \right)^2 \right\} Y_{n-1} \right. \\ \left. - B_{02} \alpha \rho Y'_{n-2} + B_{02} Y'_{n-2} - \left(A_{00} \rho_0 - B_{01} \frac{\alpha}{c_0} \right) Y_{n-2} + B_{01} Y''_{n-3} - B_{01} \alpha Y'_{n-3} \right] \quad \dots (18)$$

The solution of (17) is given by

$$Y_n = Y_n(0) \exp(\mp P_0 x) + \exp(\mp P_0 x) \int Q_n^{\pm}(x) \exp(\pm P_0 x) dx, \quad \dots (19)$$

The upper signs in the (19) represent the wave progresses in the positive direction of x and the lower signs represent the wave propagation in the negative x direction.

Now we consider a suddenly applied impulse of magnitude σ_0 at the end $x = 0$ of the filament and thereafter it is maintained steadily as, that is,

$$\sigma(0, t) = \sigma_0 H(t) \quad \dots (20)$$

where $H(t)$ is the Heavaside Unit step function.

By making use of (20) in (8), we get

$$\sigma_0 H(t) = \sum_{n=0}^{\infty} Y_n(0) F_n \{t - h(0)\} \quad \dots (21)$$

By the conditions [12],

$$Y_n(0) = \begin{cases} \sigma_0 & \text{if } n = 0 \\ 0 & \text{if } n \neq 0. \end{cases} \quad \dots (22)$$

$$h(0) = 0 \text{ and } F_0 = H(t). \quad \dots (23)$$

By using (23) and by successive integrations of (10), we get

$$F_n = \frac{\{t - h(x)\}^n}{n!} H \{t - h(x)\} \quad \dots (24)$$

Using the condition for boundary stress given by (20), the solution of (7), for the wave progresses in the positive direction of x is given by

$$\sigma(x, t) \approx \sum_{n=0}^{\infty} Y_n(x) \frac{\{t - h(x)\}^n}{n!} H \{t - h(x)\} \quad \dots (25)$$

$$h(x) = \frac{x}{c_0} \quad \dots (26)$$

where $Y_n(x)$ and $Y_n(0)$ are obtained by (19) and (22).

The approximation of first term is given by

$$\sigma(x, t) = \sigma_0 \exp(-P_0 x) H \left\{ t - \frac{x}{c_0} \right\} \quad \dots (27)$$

The equation (27) gives the expression for transient stress wave which starts from the end $x = 0$ with amplitude σ_0 and moves in the positive direction of x -axis with constant velocity c_0 . The wave is modulated by $\exp(-P_0 x)$. Other terms in the approximation solution may be obtained from (19). The solution applies until the wave moving in the positive direction of x strikes either at an interface (in the case of a composite filament) or at an end (in the case of a finite rod).

3. Special Case:

In this case we consider the density ρ obeys the law

$$\rho = \rho_0 \cos \alpha_1 x \quad \dots(28)$$

and the mechanical properties of the filament obeys

$$G_1 = G_{01} \cos \alpha_2 x, G_2 = G_{02} \cos \alpha_3 x, G_3 = G_{03} \cos \alpha_4 x \text{ and} \quad \dots (29)$$

$$\eta_2 = \eta_{02} \cos \alpha_5 x, \eta_3 = \eta_{03} \cos \alpha_6 x, \eta_{3'} = \eta_{03} \cos \alpha_7 x, \eta_4 = \eta_{08} \cos \alpha_8 x \quad \dots (30)$$

We also assume density > rigidity > viscosity.

(13) can be rewritten as

$$Y'_n + \frac{1}{2} \left\{ \rho c A_2 - (\log \rho)_{,x} - B_2 h' + (\log h')_{,x} \right\} Y_n = Q_n \quad \dots (31)$$

From (9), we have

$$\left(\frac{dh(x)}{dx} \right)^2 = \frac{\rho_0 \cos \alpha_1 x}{G_{01} \cos \alpha_2 x} = \frac{1}{c^2(x)} \quad \dots (32)$$

On integrating, we find

$$h(x) = h(0) + \int_0^x \frac{dx}{c(x)} \quad \dots (33)$$

where $c(x)$ is the velocity of the wave and is given by

$$c(x) = \sqrt{\frac{G_{01} \cos \alpha_2 x}{\rho_0 \cos \alpha_1 x}} \quad \dots (34)$$

The solution of (31) is given by

$$Y_n = Y_n(0) \sqrt{\frac{\rho(x)c(x)}{\rho(0)c(0)}} \exp \left[\mp \int_0^x \{ \rho(x)c(x)A_2(x) - B_2(x)h'(x) \} dx \right] + \int_0^x Q_n \sqrt{\frac{\rho(x)c(x)}{\rho(0)c(0)}} \exp \left[\pm \int_0^x \{ \rho(x)c(x)A_2(x) - B_2(x)h'(x) \} dx \right] dx \quad \dots (35)$$

where $h(x)$ and $c(x)$ are given by (33) and (34) respectively. Using (28), (29), and (32), $A_2(x)$ and $B_2(x)$ are obtained from (4).

For this case the first term approximation solution of (7), for wave propagated in positive direction of x axis is given by

$$\sigma(x, t) = \sigma_0 \sqrt{\frac{\rho(x)c(x)}{\rho(0)c(0)}} \exp \left[\mp \int_0^x \{ \rho(x)c(x)A_2(x) - B_2(x)h'(x) \} dx \right] H \left(t - \int_0^x \sqrt{\frac{\rho_0 \cos \alpha_1 x}{G_{01} \cos \alpha_2 x}} dx \right) \quad \dots (36)$$

(36) gives the expression for transient stress wave propagating along positive direction of

x axis with velocity $c(x) = \sqrt{\frac{G_{01} \cos \alpha_2 x}{\rho_0 \cos \alpha_1 x}}$. It is modulated by the factor.

$$\sqrt{\frac{\rho(x)c(x)}{\rho(0)c(0)}} \exp \left[- \int_0^x \left\{ \rho(x)c(x)A_2(x) - B_2(x) \sqrt{\frac{\rho_0 \cos \alpha_1 x}{G_{01} \cos \alpha_2 x}} \right\} dx \right] \quad \dots (37)$$

4. Numerical analysis and conclusion

We consider the parameters as stated in Table-1. These parameters are chosen arbitrary.

Table-1(Parameter for seven- parameter viscoelastic model)

parameter	ρ_0	G_{01}	G_{02}	G_{03}	η_{02}	η_{03}	$\eta_{03'}$	η_{04}
Assumed value	2	1.8	1.7	1.6	1.3	1.2	1.1	1.0

Using above parameters in (27), it is modulated by the factor

$$\frac{\sigma}{\sigma_0} = \exp(-4.232 - \alpha)x \tag{38}$$

and (37) at $t = h(x)$ is given by (39). Fig. 2 plotted for (38) using MATLAB represents the variation of stress ratio with distance i.e. along x- axis by taking three different values of α i.e. $\alpha = 1, 2, 3$. Fig. 3 is plotted for (39) by taking $\alpha_1 = 1, \alpha_2 = 2, \alpha_3 = 3, \alpha_4 = 4, \alpha_5 = 5, \alpha_6 = 6, \alpha_7 = 7, \alpha_8 = 8$ using MATLAB.

$$\frac{\sigma}{\sigma_0} = (\cos x \cos 2x)^{1/4} \exp \left[- \int_0^x \left\{ \frac{1.4 \cos 3x}{\sqrt{\cos x \cos 2x}} + \frac{1.46 (\cos 2x)^{1/2}}{\sqrt{\cos x}} + \frac{1.4 \cos 4x (\cos x)^{1/2}}{\cos 6x (\cos 2x)^{1/2}} + \frac{1.7 \cos 4x (\cos x)^{1/2}}{\cos 8x (\cos 2x)^{1/2}} \right. \right. \\ \left. \left. - \frac{1.45 \sqrt{\cos x \cos 2x}}{\cos 5x} + \frac{1.4 \cos 4x \sqrt{\cos x}}{\cos 6x \sqrt{\cos 2x}} + \frac{0.15 \cos 4x \sqrt{\cos x}}{\cos 7x \sqrt{\cos 2x}} \right\} dx \right] \tag{39}$$

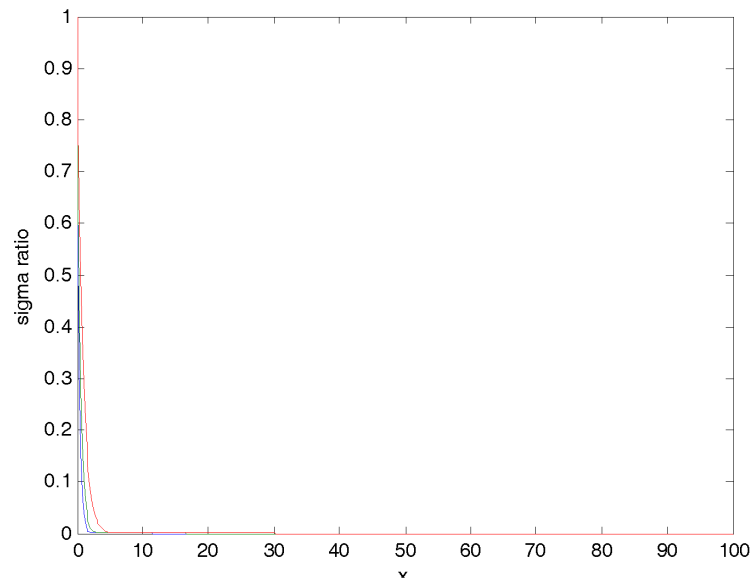


Fig.2 Variation of sigma ratio with distance x for semi-homogeneous case

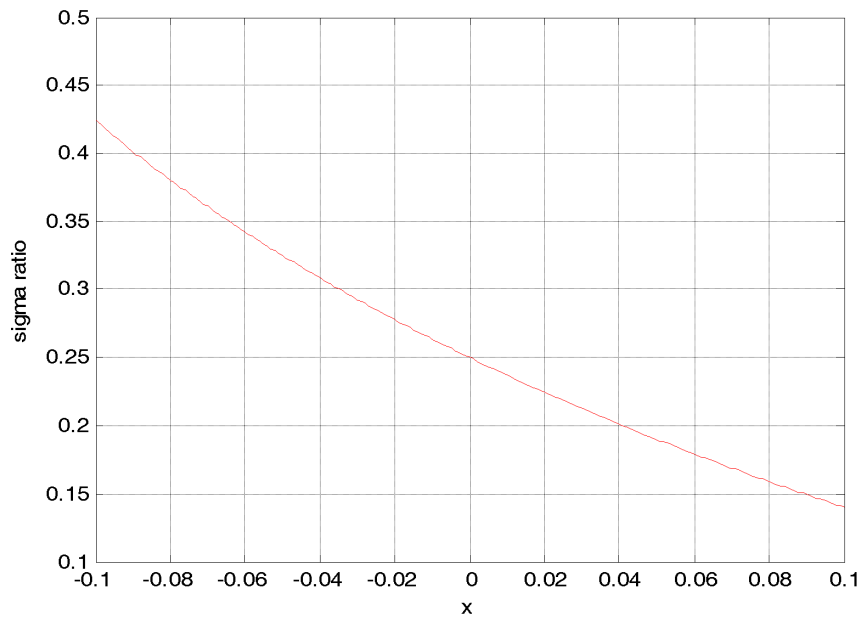


Fig.3 Variation of stress ratio with distance x in non-homogeneous case

So, it is clear from Fig. 2 that there are slight disturbances near the origin of the starting of the wave in seven parameter viscoelastic filament. The wave propagates exponentially and becomes stationary as it moves along the direction of propagation and also for all the values of α the graphs are symmetrical. The wave is moving with constant speed $c_0 = \sqrt{\frac{G_{01}}{\rho_0}}$ in the direction of x .

From Fig 3, it is observed that on passing through the distance the wave advances with depth in case of non-homogeneous parameters. As time depends upon density and rigidity of the material and therefore the amplitude of wave is going on decreasing or we can say disturbances are reduced with the passes of time and distance under stress boundary conditions for the non-homogeneous parameters. The wave advances with the speed $c(x) = \sqrt{\frac{G_{01} \cos \alpha_2 x}{\rho_0 \cos \alpha_1 x}}$ so, this model can be used for the further study in seismology.

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