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CONCIRCULAR AND PROJECTIVE CURVATURE TENSORS ON P-SASAKIAN MANIFOLD

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Abstract : In this paper we have proved the condition that if a p-Sasakian manifold M is concircularly flat, then $R(X, \xi)\cdot V = 0$ and $V(X, \xi)\cdot R = 0$. Also we have studied a Ricci-parallel p-Sasakian manifold with projective curvature tensor which satisfying the conditions $R(X, \xi)\cdot P = P(X, \xi)\cdot R$ and $R(X, \xi)\cdot P = L\{(X\wedge\xi)\cdot\rho\}$, $L \neq -1$., where R, V and ρ are the Riemannian, concircular and projective curvature tensor respectively, ξ is a characteristic vector field, L is some function on M and $X \in TM$.

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1. Preliminaries

Let M^n be n-dimensional C^∞-manifold. If there exist a tensor field F of type (1, 1), a vector field ξ and a 1-form η in M^n satisfying

$$\bar{\bar{X}} = X - \eta(X)\,\xi,\ \bar{X} = F(X), \eta(\xi) = 1 \qquad \ldots(1)$$

Then M^n is called an almost para contact manifold.

Let g be the Riemannian metric satisfying

$g(X, \xi) = \eta(X)$ ...(2)

$\eta(F, X) = 0, F\xi = 0, \text{rank } F = (n - 1)$ 1 ...(3)

$g(FX, FY) = g(X, Y) - \eta(X)\eta(Y)$ ...(4)

Then the set (F, ξ, η, g) satisfying (1), (2), (3) and (4) is called an almost para-contact Riemannian structure. The manifold with such structure is called an almost p-contact Riemannian manifold [4].

If we define $'F(X, Y) = g(\overline{X}, Y)$, then in addition to the above relations the following are satisfied:

$'F(X, Y) = 'F(Y, X)$ and $'F(\overline{X}, \overline{Y}) = 'F(X, Y)$ ...(5)

Let us consider an n-dimensional differentiable manifold M with a positive definite metric g which admits 1-forms η satisfying

$(\nabla_X\eta)(Y) - (\nabla_Y\eta)(X) = 0$ ...(6)

and $(\nabla_X\nabla_Y\eta)(Z) = -g(X, Z)\eta(Y) - g(X, Y)\eta(Z) + 2\eta(X)\,\eta(Y)\eta(Z)$ ...(7)

where, ∇ denote the covariant differentiation with respect to g. Moreover, if we put

$\eta(X) = g(X, \xi), (\nabla_X\xi) = \overline{X}$ ...(8)

Then it can be easily verified that the manifold in consideration becomes an almost para-contact Riemannian manifold. Such a manifold is called p-Saskian manifolds [1].

For a p-Saskian manifold the following relations hold:

$R(X,Y)\xi = \eta(X)Y - \eta(Y)X$ ...(9)

$R(\xi,X)Y = \eta(Y)X - g(X,Y)\xi$ ...(10)

$R(\xi,X)\xi = X - \eta(X)\xi$ ...(11)

$S(X,\xi) = -(n - 1)\eta(X)$ ...(12)

$Q\xi = -(n-1)\xi$ ....(13)

$\eta(R(X,Y)U) = g(X,U)\eta(Y) - g(Y,U)\eta(X)$ …(14)

$\eta(R(X,Y)\xi) = 0$ …(15)

$\eta(R(\xi,X)Y) = \eta(X)\eta(Y) - g(X,Y)$ …(16)

An almost paracontact Riemannian manifold M is said to be η-Einstein [5] if its Ricci tensor S is of the form

$S = ag + b\eta \otimes \xi$

where a, b are smooth functions on M.

In this case have

$S(X,Y) = ag(X, Y) + b\eta(X)\eta(Y)$ …(17)

In particular if b = 0, then M is an Einstein manifold [6]. For any vector field X, Y, Z if S is the Ricci curvature and Q is the Ricci operator then S(X, Y) = g(QX, Y) [2].

2. Concircular curvature tensor

By definition the concircular curvature tensor V is given by

$V(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y]$ …(18)

where R and r are the Riemannian curvature tensor and scalar curvature of M^n respectively.

The endomorphism $X\wedge Y$ and $X\wedge_S Y$ and the homeomorphism R(X, ξ)Vand V(X, ξ)R are defined by

$(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$ …(19)

$(X \wedge_S Y)Z = S(Y, Z)X - S(X, Z)Y$ …(20)

$(R(X, \xi)\cdot V)(U, Z)W = R(X, \xi)V(U, Z)W - V(R(X, \xi)U, Z)W$

$- V(U, R(X, \xi)Z)W - V(U, Z)R(X, \xi)W$ …(21)

$(V(X, \xi),R)(U, Z)W = V(X, \xi)R(U, Z)W - R(V(X, \xi)U, Z)W$

$$- R(U, Z(X, \xi)V)W - R(U, Z)V(X, \xi)W \qquad \ldots(22)$$

respectively, where X, Y, Z are vector fields of M[3].

Theorem 1. *A p-Sasakian manifold with concircular curvature tensor V satisfying $r = -n(n-1)$, then we have $R(X, \xi)V = 0$ and $V(X, \xi)R = 0$.*

Proof : From (21) by definition, we have

$$(R(X, \xi)\cdot V)(U, Z)W = R(X, \xi)V(U, Z)W - V(R(X, \xi)U, Z)W$$
$$- V(U, R(X, \xi)Z)W - V(U, Z)R(X, \xi)W \qquad \ldots(23)$$

For all vector fields X, Z, U, W.

Putting $U = W = \xi$ in (23) yields

$$(R(X, \xi)\cdot V)(\xi, Z)\xi = R(X, \xi)V(\xi, Z)\xi - V(R(X, \xi)\xi, Z)\xi$$
$$- V(\xi, R(X, \xi)Z)\xi - V(\xi, Z)R(X, \xi)\xi \qquad \ldots(24)$$

From (18) we get by virtue of (2) and (11),

$$V(\xi, Y)\xi = [1 + \frac{r}{n(n-1)}](Y - \eta(Y)\xi)$$

and if $r = -n(n-1)$, then $V(\xi, Y)\xi = 0$ for all vector field Y and similarly

$$V(Y, \xi)\xi = 0 \qquad \ldots(25)$$

Thus we have

$$(R(X, \xi)\cdot V)(\xi, Z)\xi = - V(R(X, \xi)\xi, Z)\xi - V(\xi, Z)R(X, \xi)\xi \qquad \ldots(26)$$

Using (11) we get

$$V(R(X,\xi)\xi, Z)\xi = -V(X, Z)\xi \qquad \ldots(27)$$

$$V(\xi,Z)R(X, \xi)\xi = - V(\xi, Z)X \qquad \ldots(28)$$

Thus we have

$$(R(X, \xi)\cdot V)(\xi, Z)\xi = V(X, Z)\xi + V(\xi, Z)X \qquad \ldots(29)$$

On the other hand

$$V(X, \xi)\cdot R)(\xi, Z)\xi = V(X, \xi)R(\xi, Z)\xi - R(V(X, \xi)\xi, Z)\xi$$

$$-R(\xi, V(X, \xi)Z)\xi - R(\xi, Z)V(X, \xi)\xi \quad \ldots(30)$$

Using (16) and (25) we obtain

$V(X, \xi)\cdot R(\xi, Z)\xi = V(X, \xi)Z$; $R(V(X, \xi)\xi, Z)\xi = 0$;

$R(\xi, V(X, \xi)Z)\xi = V(X, \xi)Z$ and $R(\xi, Z)V(X, \xi)\xi = 0$

Using these equations in (28), we have

$$(R(X,\xi)\cdot V)(\xi,Z)\xi=0 \quad \ldots(31)$$

Hence the theorem is proved.

Using the fact that a p-Sasakian manifold is concircularly flat if

$r = -n(n - 1)$, we have the following theorem:

Theorem 2. *On a concircularly flat p-Sasakian manifold, we have*

$R(X, \xi)\cdot V = 0$ *and* $V(X, \xi)\cdot R = 0$.

3. Projective curvature tensor

By definition the projective curvature tensor P is given by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[S(Y, Z)X - S(X, Z)Y] \quad \ldots(32)$$

Definition [7]. *A p-Sasakian manifold is called Ricci-parallel if* $(D_v Ric) = 0$.

which implies that $S(Y, Z) = -(n - 1)g(Y, Z)$.

Using this in (32), we have

$$P(X, Y)Z = R(X, Y)Z + [g(Y, Z)X - g(X, Z)Y] \quad \ldots(33)$$

The endomorphism $X\wedge Y$ and $X\wedge_S Y$ are defined by (20) and (21), and the homeomorphism $R(X, \xi)\cdot P$and P

$$(R(X, \xi)\cdot P)(U, Z)W = R(X, \xi)P(U, Z)W - P(R(X, \xi)U, Z)W$$

$$- P(U, R(X, \xi)Z)W - P(U, Z)R(X, \xi)W \quad \ldots(34)$$

$$(P(X, \xi),R)(U, Z)W = P(X, \xi)R(U, Z)W - R(P(X, \xi)U, Z)W$$

$$- R(U, Z(X, \xi)P)W - R(U, Z)P(X, \xi)W \qquad \ldots(35)$$

Putting $U = W = \xi$ in (34) yields

$$(R(X, \xi)\cdot P)(\xi, Z)\xi = R(X, \xi)P(\xi, Z)\xi - P(R(X, \xi)\xi, Z)\xi$$
$$- P(\xi, R(X, \xi)Z)\xi - P(\xi, Z)R(X, \xi)\xi \qquad \ldots(36)$$

From (33) we get by virtue of (2) and (11),

$$P(\xi, Y)\xi = P(Y, \xi)\xi = 0 \qquad \ldots(37)$$

for any vector field Y.

Thus we have

$$(R(X, \xi)\cdot P)(\xi, Z)\xi = -P(R(X, \xi)\xi, Z)\xi - P(\xi, Z)R(X, \xi)\xi \qquad \ldots(38)$$

Using (11) we get

$$P(R(X, \xi)\xi, Z)\xi = -P(X, Z)\xi \text{ and } P(\xi, Z)R(X, \xi)\xi = - P(\xi, Z)X$$

Thus from (37), we have

$$(R(X, \xi)\cdot P)(\xi, Z)\xi = P(X, Z)\xi + P(\xi, Z)X \qquad \ldots(39)$$

On the other hand

$$(P(X, \xi)\cdot R)(\xi, Z)\xi = P(X, \xi)R(\xi, Z)\xi - R(P(X, \xi)\xi, Z)\xi$$
$$- R(\xi, P(X, \xi)Z)\xi - R(\xi, Z)P(X, \xi)\xi \qquad \ldots(40)$$

Using (11), (14) and (37), we obtain the following equations

$$P(X, \xi)\cdot R(\xi, Z)\xi = P(X, \xi)Z \text{ ; } R(P(X, \xi)\xi, Z)\xi = 0 \text{ ;}$$

$$R(\xi, P(X, \xi)Z)\xi = P(X, \xi)Z \text{ and } R(\xi, Z)P(X, \xi)\xi = 0$$

Using these equations in (40), we have

$$(P(X, \xi)\cdot R)(\xi, Z)\xi = 0 \qquad \ldots(43)$$

Thus our condition satisfies the following equation

$$(R(X, \xi)\cdot P)(\xi, Z)\xi = 0$$

Therefore from (39), we have

$P(X, Z)\xi + P(\xi, Z)X = 0$

Using (33), (2) and (9), we have

$$R(\xi, Z)X = \eta(X)Z - g(X, Z)\xi \quad \ldots(42)$$

The above equation implies that M^n is of constant curvature -1.

Hence we can state the following theorem:

Theorem 3. *A Ricci parallel p-Sasakian manifold with projective curvature tensor satisfying the condition* $R(X, \xi)\cdot P = P\cdot(X, \xi)R$ *is of constant curvature-1 and consequently it is locally isometric to the hyperbolic space.*

We denote the expression in the bracket on the right hand side of (36) by A, and we calculate it. Thus

$$A = L\{((X\wedge\xi)\cdot P)(\xi, Z)\xi\} = L\{(X\wedge\xi)P(\xi, Z)\xi - P((X\wedge\xi)\xi, Z)\xi$$

$$- P(\xi, (X\wedge\xi)Z)\xi - P(\xi, Z)(X\wedge\xi)\xi\} \quad \ldots.(41)$$

Using (37) we have

$(X\wedge\xi)P(\xi, Z)\xi = 0$; $P((X\wedge\xi)\xi, Z)\xi = P(X, Z)\xi$;

$P(\xi, (X\wedge\xi)Z)\xi = 0$ and $P(\xi, Z)(X\wedge\xi)\xi = P(\xi, Z)X$

From the above and using (39), we have

$$(L + 1)\{P(X, Z)\xi + P(\xi, Z)X\} = 0 \quad \ldots(44)$$

Using (2), (9) and (33), we have

$(L + 1)\{R(\xi, Z)X + g(X, Z)\xi - \eta(X)Z\} = 0$

Since $L \neq -1$, we get

$$R(\xi, Z)X = \eta(X)Z - g(X, Z)\xi \quad \ldots(45)$$

The above equation implies that M^n is of constant curvature -1.
Hence we can state the following theorem:

Theorem 4. *A Ricci-parallel p-Sasakian manifold M^n with projective curvature tensor $R(X, \xi)\cdot P = L\{(X\wedge\xi)\cdot P\}$, $(L \neq -1)$ is of constant curvature -1 and consequently it is locally isometric to the hyperbolic space.*

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