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SKEW PRODUCT FLOW ON SOME FIBRE BUNDLES

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Abstract : In this paper we study skew product flow on locally trivial fibre bundle and non trivial fibre bundle and prove some results on distal properties in this bundles over a Banach manifold.

Keywords : Skew product flow, fibre bundle, distal property.

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1. Introduction

Skew product flow was originated in the study of ordinary differential equation, introduced by Sell and Sacker in [5], on the product space $W = X \times Y$ of two topological spaces X and Y. It was further studied by Miller [2], Miller and Sell [3], Sell and Sacker [6] and Egawa [1]. In [5] authors defined distality of projection map and they obtained equivalent criteria of distal property of projection map. In this paper we have defined skew product flow on fibre bundle over a Banach manifold and also obtained local representation of such flow. Then we have defined distality of projection map on compact $\pi -$ invariant subset of the fibre bundle and have obtained equivalent criteria of distality of projection map on it.

2. Preliminaries

If W denote a topological space and T denote a set of real numbers R or the set of intergers Z, then a mapping π is defined by

$\pi : W \times T \to W$ is called a flow [5] if

i) π is continuous,

ii) $\pi(\omega,0)=\omega$,

iii) $\pi(\pi(\omega,s),t)=\pi(\omega,s+t),\omega\in W$ and $s,t\in T$.

If $T=R,\pi$ is called a continuous flow. If $T=Z,\pi$ is called a discrete flow.

A set $M\subseteq W$ is said to be $\pi-$invariant if $\gamma(\omega)\subseteq M$ whenever $\omega\in M$ where M is positively or negatively $\pi-$invariant if $\gamma^{+}(\omega)\subseteq M$ whenever $\omega\in M$ where $\gamma^{+}(\omega)=\{\pi(\omega,t):t\in R,t\geq 0\}$

or $\gamma^{-}(\omega)\in M$ where $\gamma^{-}(\omega)\subseteq\{\pi(\omega,t):t\in R,t\leq 0\}$.

Let X and Y be topological spaces and let T = R or Z. A flow, $\pi:X\times Y\times T\to X\times Y$ is called a skew product flow on X × Y [4] if there exits a continuous mapping ϕ on X × Y such that $\phi:X\times Y\times T\to X$ and a flow σ on Y where $\sigma:Y\times T\to Y$, π has the form (ϕ, σ) where $\pi(x,y,t)=(\phi(x,y,t),\sigma(y,t))$ $\forall$ $x\in X,y\in Y,t\in T$. Here we shall consider T =R throughout our discussion.

In [6] the authors also defined distal property of the projective map in the following way.

Let $\pi=(\phi,\sigma)$ be a skew product flow on $X\times Y$ and $M\subseteq X\times Y$ be a compact $\pi-$invariant set with $p:X\times Y\to Y$ be the natural projection and let $\mu(y)$=card $(p^{-1}(y)\cap M)$, for $y\in Y$ and X be also metrizable then p/M is said to be of distal type(or M is said to have the fibre distal property) if for any two points (x_1,y) and (x_2,y) in $p^{-1}(y)\cap M$ with $x_1\neq x_2$, there is an $\alpha=\alpha(x_1,x_2,y)>0$ such that

$$d(\phi(x_1,y,t),\phi(x_2,y,t))>\alpha,\ t\in R, \quad ....(1)$$

where d is the metric of the topological space X. If (1) holds for $t\geq 0$ then p/M is said to be of positive distal type and if (1) holds for $t\leq 0$ then p/M is said to be of negative distal type.

A set $M \subseteq X \times Y$ is said to be an N– fold covering space of Y with the covering projection p, if Card $p^{-1}(Y) = N,\ 0 < N < \infty$ and for each $y \in Y$, there is an open neighbourhood V of y such that $p^{-1}(V)$ consist of N disjoint open sets U_i and $p/U_i : U_i \to V$ is a homeomorphism for each $i = 1,2,....N$.

Sell and Sacker have proved the structure theorem in [6] which is given below.

Theorem 2.1: *Suppose π be a skew-product flow on X×Y where Y is a compact minimal set (where minimal set is a closed invariant set which contains no proper subset with the same properties) in the flow σ and X is metrizable. Let M be a compact π-invariant subset of X × Y, Then the following statements are equivalent.*

(A) p/M is of distal type and there is a $y_0 \in Y$ such that $\mu(y_0) = N$ is finite.

(B) p/M is of positive(or negative) distal type and there is a $y_0 \in Y$ such that $\mu(y_0) = N$ is finite.

(C $\mu(y) = N < \infty$ for all $y \in Y$.

(D) M is an N–fold covering space of Y with covering map p/M.

Also if any one of the above conditions is satisfied then M can be expressed as a finite union of minimal sets $M_1,..M_k$, where M_i are n_i -fold covering of $Y, i = 1,2,..k$ and $n_1 + n_2 + ... + n_k = N$.

While proving the theorem 2.1 the authors obtained some observations, which has important role in our paper.

Observation 2.1: If $\mu(y) = N < \infty$ and (C) holds for all $y \in Y$, then there is a $\beta > 0$ such that for all $y \in Y$ and any pair $\omega_1, \omega_2 \in p^{-1}(y)$ with $\omega_1 \neq \omega_2$. $d.\rho(\omega_1.t, \omega_2.t) = d(q(\omega_1.t), q(\omega_2.t)) \geq 3\beta$, for every pseudometric ρ and all $t \in R$ where $q : X \times Y \to X$ such that $q(x, y) = x$. In other words, if (C) holds, then *p/M* is of distal type and then $\alpha = \alpha(x_1, x_2, y)$ given in the definition of distal type can be chosen uniformly equal to 3β .

Observation 2.2 When (C) holds, p/M becomes an open mapping. In fact, for every convergent net $\{y_n\}$ in *Y* with $y_n \to y$,

lim sup $p^{-1}(y_n) = p^{-1}(y)$ and *p/M* is open.

Our objectives is to study skew product flow on a locally trivial fibre bundle.

3. Skew product flow on a locally trivial fibre bundle

In this section we state the definitions of locally trivial fibre bundle and skew product flow on locally trivial fibre bundle and discuss distality on this space accordingly Sell and Sacker [5], [6].

Definition 3.1 *A topological space W is a locally trivial fibre bundle with base space Y, fibre X and projection p, if*

(i) W, X, Y *are all topological spaces.*

(ii) $p: W \to Y$ *is a continuous mapping of W onto Y.*

(iii) For each $y \in Y$ *there is a neighbourhoood U of y such that* $p^{-1}(y)$ *is homeomorphic to the product space* $X \times U$ *i.e, there exists a homeomorphism,* $\phi : X \times U \to p^{-1}(y)$ *such that for each* $y \in U$*, the restriction of* ϕ *to* $X \times \{y\}$ *is a homeomorphism of* $X \times \{y\}$ *onto* $p^{-1}(y)$.

Definition 3.2 *Let W be a locally trivial fibre bundle with compact base space Y, fibre X and a projection p. A flow* π *on W is called a skew-product flow, if there exists a flow* σ *on Y such that* $p(\pi(w,t)) = \sigma(p(w),t) w \in W, t \in R$

In other words, π *is a skew-product flow, if p commutes with the flows* π *and* σ.

Result 3.1 *A skew product flow* π *on W has a local representation of the form* $\pi(x,y,t)=(\overline{\phi}(x,y,t),y.t)$, *where* $y.t=\sigma(y,t)$ *where* σ *is a flow on* Y. $\overline{\phi}(x,y,t)$ *is a continuous map from* $X\times U$ *to* X,U *varying over a collection of small neighbourhoods of Y.*

Proof: Let U be an open set in Y with the property that both $p^{-1}(U)$ and $p^{-1}(U.t)$ are homeomorphic to $X\times U$ and $X\times U.t$ respectively. Let $\phi : p^{-1}(U)\to X\times U$ and $\phi^{t} : p^{-1}(U.t)\to X\times U.t$ be the homeomorphisms, such that $(x,y)\in X\times U, w=\phi^{-1}(x,y)\in p^{-1}(U)\subseteq W$ and $\pi(w,t)\in p^{-1}(U.t)$, $\pi_t(\pi(w,t))\in X\times U.t$ and $\phi^{t}(\pi(w,t))=(\overline{\phi}(x,y,t),\sigma(y,t))$, where $\overline{\phi}(x,y,t)=\phi^{t}\circ\pi_t\circ\phi^{-1}(x,y)$ is a mapping from W to X defined for all $y\in U$ and $t\in R$, where $\phi^{t}=\pi(w,t)$. Thus we get the result.

Definition 3.3 *Let* π *be a skew product flow on a locally trivial fibre bundle W, with compact base space Y, fibre X and projection p. For this, we take M be a compact* π *-invariant set in W, where W is metrizable. Then p/M is of distal type(or M has the fibre distal property) if for,* $w_1,w_2\in p^{-1}(y)\cap M, w_1\neq w_2$, *there is an* $\alpha=\alpha(w_1,w_2)>0$ *such that,* $d(\pi(w_1,t),\pi(w_2,t))\geq\alpha,\forall t\in R$. *d being the metric on W.*

Now we state and prove a theorem as follows

Theorem 3.1 *If* π *be a skew-product flow on a locally trivial fibre bundle W, where* W,X,Y *are metrizable and Y is compact minimal with respect to the flow* σ *and if M be a compact* π*-invariant set in W then the following statements are equivalent*

(A) p/M is of distal type and for some $N, 0<N<\infty$, $\mu(y_0)=Card\{p^{-1}(y_0)\cap M\}=N$ *for some* $y_0\in Y$.

(B) p/M is of positive(negative) distal type and for $y_0 \in Y, \mu(y_0) = N, 0 < N < \infty$.

(C) $Card\{p^{-1}(y) \cap M\} = N, \ \forall y \in Y$ *for some* $N, 0 < N < \infty$.

(D) M is an N-fold covering of Y.

Finally if any of these conditions hold then M can be expressed as the disjoint union of $M_1, \ldots M_k$ *of compact minimal sets where each* M_i *is an* n_i *fold covering of Y and* $n_1 + n_2 + \ldots + n_k = N$.

Proof: As W, X, Y are metrizable, the proof follows from Theorem 2.1 of [5].

4. Skew product flow on a fibre-bundle over a Banach Manifold.

Firstly we define Banach manifold, fibre bundle over a Banach manifold and skew product flow on it. In this section we have obtained local representation of skew-product flow on W.

Next such local representation of skew product flow on a fibre bundle has been used to define distality of the projection map. At last by applying equivalent local criteria of distality corresponding equivalent criteria of distal map on a fibre bundle over a Banach manifold has been obtained.

Definition 4.1 *A Banach manifold B is a topological space such that every point has a neighbourhood, homeomorphic to an open set of a Banach space possessing* c^{∞} *structure.*

Example: We can consider C^n, R^n as Banach manifold.

Definition 4.2 *A fibre bundle whose base space is a Banach manifold is called a fibre bundle over a Banach manifold.*

Definition 4.3 *If W is a fibre bundle with a compact base space B and projection p then a flow* π *on W is called a skew product flow, if there exists a flow* σ *on B such that*

$$p(\pi(w,t)) = \sigma(p(\omega),t);\ \omega \in W,\ t \in R. \qquad ....(2)$$

Now we prove a theorem as follows

Theorem 4.*1 Let* π *be a skew product flow on a fibre bundle* (p,W,B) *with base space B (a real Banach manifold) and projection p and let the corresponding flow on B be* σ*. Let* $b_1 \in B_1$ *and* $b_1.t = \sigma(b_1,t) \in B_1.t; B_1, B_1.t$ *being open in B and let* $p^{-1}(b_1)$ *and* $p^{-1}(b_1.t)$ *be diffeomorphic to local bundles* $\{u_1\} \times E_1$ *and* $\{u_1.t\} \times F_1$ *where* $u_1 \in U_1$ *respectively (*E_1, F_1 *being real Banach spaces) under* ϕ *and* ψ*, where* $\phi : p^{-1}(B_1) \to U_1 \times E_1$ $\psi^t : p^{-1}(B_1.t) \to U_1.t \times F_1$.

Then π *has a local representation of the form* $\pi_1 = (\sigma', \phi')$ *such that* $\pi(u_1,e_1,t) = (\sigma'(u_1.0,t), \phi'(u_1,e_1,t)), e_1 \in E_1$ *where* $\phi' = \psi^t \circ \pi_t \circ \phi^{-1}$ *is a* c^∞ *map from* E_1 *to* F_1 *for each* $u_1 \in U_1'$*,* U_1' *varying over small neighbourhoods of* u_1 *in* U_1 *at some time t and* σ' *is such that* $\sigma(\phi(b_1).0,t) = \psi^t \circ \sigma'(b_1,t)$ *for each* $b_1 \in B_1$ *(i.e,* σ' *is a local flow on* $U_1 \times \{0\}$*)*

Proof: For each $(u_1,e_1) \in U_1 \times E_1, w_1 = \phi^{-1}(u_1,e_1) \in p^{-1}(B_1) \subset W_1 \subset W$. Also $\pi(w_1,t) \in p^1(B_1.t) \subset W$. So $\psi^t(\pi(w_1,t)) \in U_1.t \times F_1$. As π is skew product flow, p commutes with π and σ. i.e, $p(\pi(w_1,t)) = \sigma(p(w_1),t)$. Hence $\pi(w_1,t) = p^{-1}(\sigma(p(w_1),t))$.

So $\psi^t(\pi(w_1,t)) = \psi^t(p^{-1}(b_1.t)) \in U_1.t \times F_1$. i.e, local representation of π has the form $\pi_1 = (\sigma', \phi')$ which is given by

$$\pi_1(u_1,e_1,t) = (\sigma'(u_1.0,t), \phi'(u_1,e_1,t)), e_1 \in E_1$$

where $\phi'(u_1, e_1, t) = \psi^t \circ \pi_t \circ \phi^{-1}(u_1, e_1)$ is a c^∞ map from E_1 to F_1 for each $u_1 \in U_1'$, U_1' varying over all small neighbourhoods of u_1 in U_1 and some time t.

Definition 4.4 *Let W be a fibre bundle with projection p and base B which is a Banach manifold.*

Let σ be a flow on *B* and let (σ_i', ϕ_i') be the local representations of a skew product flow π on local bundles $U_i \times E_i$, $i = 1,2,3,\ldots$ (E_i being Banach Spaces) Let $M \subset W$ be a compact π – invariant subset and $B' \subset B$ be a totally bounded subset which is minimal with respect to σ. Then, *p/M* is defined to be of distal type (or *M* is said to have the fibre distal property) if for any two points $w_i \neq w_{i'}$ on *W*, $w_i, w_{i'} \in p^{-1}(b_i) \cap M$, $b_i \in B'$, there is an $\alpha_i = \alpha_i(e_i, e_{i'}, u_i) > 0, i = 1,2,3,\ldots$, such that

$$P\phi_{i'}(u_i, e_i, t) - \phi_{i'}(u_i, e_{i'}, t)P_{F_i} \geq \alpha_i, \quad t \in R \qquad \text{....(3)}$$

where $p^{-1}(b_i)$ is diffeomorphic to $\{u_i\} \times E_i$ and $p^{-1}(b_i.t)$ is diffeomorphic to $u_i.t \times F_i$, where $e_i, e_i' \in E_i$, ϕ' is a c^∞ from E_i to F_i for each $u_i \in U_i'$ where U_i' is varying over all small neighbourhoods of u_i in U_i and some time *t* and $\|.\|_F$ is denoted by $P.P_F$. Here *p/M* is said to be of positively (or negatively) distal type according as (3) holds only for $t \geq 0$ (or $t \leq 0$).

Definition 4.5 *Let* (p, W, B) *be a fibre bundle over a Banach manifold B. A set* $M \subseteq W$ *is said to be a finite fold covering space of B if,*

(i) for each $b \in B$, *Card* $\{p^{-1}(b) \cap M\}$ *is finite.*

(ii) If, Card $\{p^{-1}(b) \cap M\} = N$, $0 < N < \infty$, *then there is an open neighbourhood V of b such that* $p^{-1}(V) \cap M$ *consists of N disjoint open sets* U_i, *where* $p_i : U_i \to V$ *is a homeomorphism, for each* $i = 1,2,3,\ldots,N$.

We now prove the following structure theorem

Theorem 4.2 *Let π be a skew product flow on a fibre bundle W, with projection p and base space B, which is a Banach manifold. Let σ be the flow on B and let $B' \subset B$ be a totally bounded subset which is minimal with respect to the flow σ. Then the following statements are equivalent*

(A) p/M_0 *is of distal type and there is a* $b_0 \in B'$ *such that* $\mu_0(b_0) = N_0 < \infty$, *for some* $\pi -$*invariant subset* M_0 *of W where* $\mu_0(b_0) = Card\{p^{-1}(b_0) \cap M_0\}$.

(B) p/M_0 *is of positive(negative) distal type and there is a* $b_0 \in B'$ *such that* $\mu_0(b_0) = N_0$ *is finite.*

(C) *For each* $b \in B', \mu'(b)$ *is finite, where* $\mu'(b) = Card\{p^{-1}(b) \cap M'\}$, *for some* $\pi -$*invariant subset* M' *of W.*

(D) *There is a finite fold covering space of* B'.

Proof: We see that obviously $(A) \Rightarrow (B)$.

To show $(C) \Rightarrow (A)$, let us take any two points $\omega_1, \omega_1' \in W, \omega_1 \neq \omega_1'$, such that $\omega_1, \omega_1' \in p^{-1}(B_1), B_1$ being open in B'.

Let $\phi_1 : p^{-1}(B_1) \to U_1 \times E_1$ and $\psi_1^t : p^{-1}(B_1.t) \to U_1.t \times F_1$ be diffeomorphisms, where $U_1.t = \sigma_1'(U_1.0, t)$, σ_1' being given by

$\sigma_1'(\phi_1(b_1).0, t) = \psi_1^t \circ \sigma(b_1, t),\ b_1 \in B_1; E_1, F_1$ being Banach spaces.

Let M' be some compact $\pi -$invariant subset of W.

Let us consider $M_1 \subseteq p^{-1}(b_1) \cap M'$, such that it is also $\pi -$invariant.

Let $\phi_1(M_1) = M_1'$ (say) $\subset U_1 \times E_1$.

Now by (C), $\mu'(b_1) = N' < \infty$, for some $N' > 0$, where

$\mu'(b_1) = \text{Card}\{p^{-1}(b_1) \cap M'\}$.

So M_1' also contains finite number of elements.

Let $q_1 : U_1 \times E_1 \to U_1 \times 0$ be the natural projection.

Then $q_1^{-1}(u_1,0) \cap M_1'$, $(u_1,0) \in U_1 \times \{0\}$ contains finite number of elements. Again we know that π has a local representation of the form (σ_1', ϕ_1') on $U_1 \times E_1$, where σ_1' is the induced flow on U_1' and $\phi_1' = \psi_1^t \circ \pi_t \circ \phi_1^{-1}$ is a c^∞ map from E_1 to F_1 for each $u_1 \in U_1$ and for some $t \in R$. So, M_1' remains (σ_1', ϕ_1') invariant in $U_1 \times E_1$.

Thus condition (C) of Theorem-2.1 holds, hence condition (A) of Theorem-2.1 follows. So q_1/M_1 is of distal type. Hence there exists an $\alpha_1 = \alpha_1(u_1, e_1, e_1') > 0$ such that, $P\phi_1'(u_1,e_1,t) - \phi_1'(u_1,e_1',t)P_{F_1} \geq \alpha_1$, for each $u_1 \in U_1$, $e_1, e_1' \in E_1$ and for some $t \in R$.

Similarly, for $B_2 \subset B', B_2$ being open, if $p^{-1}(B_2)$ is diffeomorphic to $U_2 \times E_2, p^{-1}(B_2.t)$ is diffeomorphic to $U_2.t \times F_2$ and if

$M_2 \subseteq p^{-1}(b_2) \cap M', b_2 \in B_2$ be $\pi-$invariant, then we have,

$$P\phi_2'(u_2,e_2,t) - \phi_2'(u_2,e_{2'},t)P_{F_2} \geq \alpha_2 > 0,$$

for each $u_2 \in U_2, e_2, e_2' \in E_2$ and for some $t \in R, \alpha_2 = \alpha_2(u_2, e_2, e_2')$.

Now if we consider $\alpha' = min(\alpha_1, \alpha_2) > 0$ and $M_0' = M_1 \cup M_2$ then $P\phi_i'(u_i,e_i,t) - \phi_i'(u_i,e_i',t)P_{F_i} \geq \alpha'$, for each $u_i \in U, U = U_1 \cup U_2, e_i, e_i' \in E_i$ and for some $t \in R, i = 1,2.$

Continuing this process and assuming $M = \bigcup_i M_i$, M becomes a compact $\pi -$ invariant subset of *W*.

Also for any $b_i \in B' = \bigcup_i B_i$ and $\omega_1, \omega_2 \in p^{-1}(b_i) \cap M$, there is an $\alpha = \inf_i\{\alpha_i\}$ such that $P\phi_i'(u_i, e_i, t) - \phi_i'(u_i, e_i', t)P_{F_i} \geq \alpha$,

for some $i, i = 1,2,..u_i \in U = \bigcup_i U_i; e_i, e_i' \in E_i$ and for some $t \in R$. Taking $M = M_0$, it follows that p/M_0 is of distal type.

The 2nd part follows from (C) by taking $b_0 = b$. Thus proof of $(C) \Rightarrow (A)$ is complete.

Now we have to show $(B) \Rightarrow (C)$, and so let $b_0 \in B' \subset B$ be given with $\mu_0(b_0) = N_0 < \infty$ and let *b* be any other point in B'.

Since B' is minimal, i.e, B' is non-empty, closed and $\sigma -$ invariant, the set $\gamma^+(b_0) = \{\sigma(b_0, t), t \in I_{b_0}, t \geq 0\}$ is dense in B'.

Therefore, there is a sequence $\{t_n\}$ in *R* such that

$b_0.t_n \to b$, where $b_0.t_n = \sigma(b_0, t_n)$.

Since $\mu_0(b_0) = N_0 < \infty$, let $p^{-1}(b_0) \cap M = \{\omega_1, \omega_2, \ldots, \omega_{N_0}\}$, M being compact $\pi -$ invariant in *W*.

Now let $p^{-1}(b_0)$ be diffeomorphic to $\{u_0\} \times E_0$ and $p^{-1}(b_0.t)$ be diffeomorphic to $\{u_0.t\} \times F_0$ under ϕ_0 and $\psi_0{}^t$, and let (σ_0', ϕ_0') be a local representation of π.

By positive distal property of the projection map *p*, there is an $\alpha = \alpha(u_0, e_1, e_2)$ such that

$$P\phi_0'(u_0,e_0,t)-\phi_0'(u_0,e_{0'},t)P_{F_0} \geq \alpha, \qquad \text{....(4)}$$

for each $u_0 \in U_0; \omega_i, \omega_j \in p^{-1}(b_0) \cap M$, $\omega_i \neq \omega_j, i, j = 1,2,...N_0$,

$e_0, e_0' \in E_0$ and for some $t \geq 0$.

We choose subsequences $\{\omega_1.t_n\},...,\{\omega_{N_0}.t_n\}$ where $\pi(\omega_i,t_n)=\omega_i.t_n$, $i=1,...,N_0$.

Then they will converge to limits, say $\hat{\omega}_1,...,\hat{\omega}_{N_0}$ (as p commutes with the flow and $b_0.t_n \to b$ in B'). So, $\hat{\omega}_1,\hat{\omega}_2,\hat{\omega}_{N_0} \in p^{-1}(b)$.

Now from (4.3), $P\phi_0'(u_0,e_0,t_n)-\phi_0'(u_0,e_0',t_n)P_{F_0} \geq \alpha$, $t_n \geq 0$.

Taking limits as $n \to \infty$ and considering continuity of $\mathsf{P}\cdot\mathsf{P}_{F_0}$, we have

$$\mathsf{P}\phi'(u,e,t)-\phi'(u,e',t)\mathsf{P}_F \geq \alpha, \ u \in U, \ \ e,e' \in E,$$

where $\phi: p^{-1}(b) \to U \times E$, $\psi^t : p^{-1}(b.t) \to U.t \times F$ are diffeomorphisms, $\phi' = \psi^t \circ \pi_t \circ \phi^{-1}$ is c^∞ map from (E to F) for each $u \in U$, $t \in R$.

So, taking $q: U \times E \to U \times \{0\}$, as the natural projection,

Card $(q^{-1}(u,0)) \geq N_0$. Now we show that, Card $(q^{-1}(u,0)) = N_0$.

If possible, let Card $(q^{-1}(u,0)) = N_0 + 1$. Then considering $b.t_n \to b_0$ in B' and arguing similarly as above, we shall get

$$\text{Card}\,(q^{-1}(u_0,0)) \geq N_0 + 1$$

and hence $\mu(b_0)$=Card $\{\phi^{-1}(q^{-1}(u_0,0)) \geq N_0 + 1$,

which is a contradiction to the fact that $\mu(b_0) = N_0$.

So $\mu(b) = N_0 < \infty$ for $b \in B'$.

Thus we see that (C) follows from (B) where $N' = N$ and $M' = M$.

Next we have to show $(C) \Rightarrow (D)$.

Let us fix an element $b \in B_1 \subset B', B_1$ being open.

Let $\phi : p^{-1}(B_1) \to U \times E$ and $\psi^t : p^{-1}(B_1.t) \to U.t \times F$ be diffeomorphisms, where E, F being Banach spaces.

As $\mu'(b)$ is finite i.e, Card $\{p^{-1}(b) \cap M'\}$ is finite, for some π – invariant subset M' of *W*, so taking $M_1 \subseteq p^{-1}(b) \cap M'$, as π – invariant in W, $\phi(M_1)$ also contains finite number of elements.

Let $q : U \times E \to U \times \{0\}$ be the natural projection.

Then $\text{Card}(q^{-1}(U \times \{0\}) \cap \phi(M))$ is finite. Let these elements be $e_1, e_2, ..., e_N$. By theorem 2.1, $q/\phi(M_1)$ is of distal type.

Thus condition (C) of theorem-2.1 holds. Now by observation-2.2 of theorem-2.1, q becomes open. Further from observation-2.1 of the same theorem, there is a $\beta > 0$ such that for each $u \in U$, and for any pair of elements $e_i, e_j \in \phi(M_1),\ i \neq j$.

$$\|\phi'(u, e_i, t) - \phi'(u, e_j', t)\|_F \geq 3\beta \quad \text{for some } t \in R.$$

So, proceeding as in theorem 2.1, it is found that there exists some $0 < N' < \infty$ such that $\phi(M_1)$ is an N' – fold covering space of *U* with covering map $q/\phi(M_1)$. i.e, for each $(u,0) \in U \times \{0\}$ there is an open neighbourhood $(u,0) \in U \times \{0\}$ such that $q^{-1}(U \times \{0\}) \cap \phi(M_1)$ consists of N' disjoint open sets $U_i \times E_i$ where $q_i : U_i \times E_i \to U \times \{0\}$ is a homeomorphism for each $i = 1, 2, ..., N'$.

Then $\phi^{-1}(U_i \times E_i) = \overline{U_i}$ are disjoint open sets in $p^{-1}(B_1) \cap M'$ for $i = 1, 2, ..., N'$.

As p commutes with the flows, so $p_i = \phi^{-1} \circ q_i \circ \phi$ becomes homeomorphisms from $\overline{U_i}$ to open neighbourhood B_1 of b.

Thus from definition-4.5, M' is a finite fold covering space of B_1.

This completes the proof of the Theorem 4.2.

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References

[1] Egawa, Jiro (1987) *Stability and skew-product flows,* Proc. American Math. Soc., Memoir No. **101,** 484-486.

[2] Miller R.K. (1965) *Almost periodic differential equations as dynamical systems with applications to the existence of AP solutions,* Jour. Diff. Eqs. **1,** 337-345.

[3] Miller R.K., Sell G.R (1970) *Volterra integral equations and topological dynamics,* American Math. Soc., Memoir No. **102,** 1-69.

[4] Sacker R.J , Sell G.R (1973) *Skew-product flows, finite extensions of minimal transformation groups and almost periodic differential equations.* Bull American Math. Soc., **79,** 802-805

[5] Sacker R. J, Sell G. R (1977) *Lifting properties in skew-product flows with applications to differential equations,* American Math. Soc., Memoir No. **190**(11), 1-67.

[6] Sell G.R and Sacker R.J (1974) *Finite extensions of minimal transformation groups,* Trans. of American Math. Soc., **190,** 325-334.