Journal of Rajasthan Academy of Physical Sciences
ISSN : 0972-6306; URL : http://raops.org.in
Vol. 12, No. 4, Dec. 2013, pp 385-392

GENERALIZED EULER POLYNOMIALS AND THEIR PROPERTIES

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Received : Oct. 2, 2013

Abstract : The Euler polynomials $E_n(x)$ and the Euler number E_n play an important role in various fields like analysis, number theory, differential geometry and algebraic topology. The aim of this paper is to derive some interesting properties of generalized Euler polynomials.

Keywords : Euler number, Euler polynomials, generalized Euler polynomials, generating function.

2010 Mathematics Subject Classification : 2010-11B68

1. Introduction

For a real or complex parameter α, the generalized Euler polynomials $E_n^{\alpha}(x)$ of degree n in x as well as in α are defined by the following generating function [3,4] :

$$\frac{(2)^{\alpha} e^{xt}}{(e^t+1)^{\alpha}} = \sum_{n=0}^{\infty} E_n^{\alpha}(x) \frac{t^n}{n!} \quad \left[|t| \leq \pi\right] \qquad ....(1)$$

2. A recurrence relation for $E_n^\alpha(x)$

In this section, we derive the following recurrence relation for $E_n^\alpha(x)$:

$$E_n^\alpha(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^\alpha 2^{-k} (x - \frac{1}{2})^{n-k} \qquad \text{....(2)}$$

Proof : By (1) we get

$$\frac{(2)^\alpha \left[1 + xt + \frac{x^2t^2}{2!} + ... + \frac{x^nt^n}{n!}\right]}{\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + ... + 1\right]^\alpha} = \sum_{n=0}^{\infty} E_n^\alpha(x) \frac{t^n}{n!}$$

or $$\left[1 + xt + \frac{x^2t^2}{2!} + ... + \frac{x^nt^n}{n!}\right] = \left[1 + \frac{t}{2^\alpha} + \frac{t^2}{2^\alpha.2!} + \frac{t^3}{2^\alpha.3!} + ...\right]^\alpha \sum_{n=0}^{\infty} E_n^\alpha(x) \frac{t^n}{n!}$$

or $$\left[1 + xt + \frac{x^2t^2}{2!} + ... + \frac{x^nt^n}{n!}\right] = \left[1 + \frac{t}{2^\alpha} + \frac{t^2}{2^\alpha.2!} + \frac{t^3}{2^\alpha.3!} + ...\right]^\alpha$$

$$\left[E_0^\alpha(x)\frac{t^0}{0!} + E_1^\alpha(x)\frac{t^1}{1!} + E_2^\alpha(x)\frac{t^2}{2!} + ...\right]$$

Using Binomial expansion, we get

$$\left[1 + xt + \frac{x^2t^2}{2!} + ... + \frac{x^nt^n}{n!}\right] = \left[1 + \left(\frac{\alpha t}{2^\alpha} + \frac{\alpha t^2}{2^\alpha.2!} + \frac{\alpha t^3}{2^\alpha.3!} + ..\right)\right.$$

$$\left. + \frac{\alpha(\alpha-1)}{2!}\left(\frac{t}{(2^\alpha)^2} + \frac{t^2}{(2^\alpha)^2.4} + \frac{t^3}{(2^\alpha)^2.36} + ..\right) + ...\right]$$

$$\left[E_0^\alpha(x)\frac{t^0}{0!} + E_1^\alpha(x)\frac{t^1}{1!} + E_2^\alpha(x)\frac{t^2}{2!} + ...\right]$$

Comparing the coefficients of various powers of *t*, we get

$$E_0^\alpha(x) = 1, \quad E_1^\alpha(x) = x - \frac{\alpha}{2^\alpha},$$

and $E_2^\alpha(x) = x^2 - \frac{2\alpha x}{2^\alpha} - \frac{2\alpha^2}{2^\alpha.2^\alpha} - \frac{\alpha}{2^\alpha} - \frac{\alpha(\alpha-1)}{2^\alpha.2^\alpha}$ and so on.

Clearly, we have

$$E_n^{(1)}(x) = E_n(x)$$

Again the equation (1) can be written as

$$\frac{(2)^{\alpha}.e^{t/2.}.e^{t(x-\frac{1}{2})}}{(e^t+1)^{\alpha}} = \sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n}{n!}$$

We also know that the generating function of generalized Euler number is

$$\frac{(2)^{\alpha}.e^{t}}{(e^{2t}+1)^{\alpha}} = \sum_{n=0}^{\infty} E_n^{\alpha}\frac{t^n}{n!}$$

Therefore

$$\sum_{n-0}^{\infty} E_n^{\alpha}\frac{t^n}{n!}.\frac{1}{2^n}.\sum_{n=0}^{\infty}(x-\frac{1}{2})^n\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n}{n!}$$

By the Cauchy product rule for multiplying power series, we easily arrive at

$$E_n^{\alpha}(x) = \sum_{k=0}^{n}\binom{n}{k}E_k^{\alpha}2^{-k}(x-\frac{1}{2})^{n-k}$$

Particular cases :

(i) Put x=1/2 in (2) we get the generalized Euler number [2]

$$E_n^{\alpha}(1/2) = 2^{-n.}E_n^{\alpha} \text{ and } E_n^{\alpha} = 2^{n.}E_n^{\alpha}(1/2) \qquad ....(3)$$

(ii) Put α=1 in (3), we get the ordinary Euler number E_n as

$$E_n = 2^n E_n(1/2) \qquad ....(4)$$

(iii) For α=1, (2) reduces to the well-known recurrence relation of Euler polynomials :

$$E_n(x) = \sum_{k=0}^{n}\binom{n}{k}E_k\,2^{-k}(x-\frac{1}{2})^{n-k} \qquad .....(5)$$

3. Properties of generalized Euler polynomials

In this section, some interesting properties of generalized Euler polynomials are derived by using the generating function (1).

Property 1 (Differentiation formula) :

$$\frac{d}{dx}\left[E_n^{\alpha}(x)\right]=n.E_{n-1}^{\alpha}(x) \qquad ....(6)$$

Proof : Differentiating equation (1)with respect to *x* both sides, we get

$$\frac{2^{\alpha}e^{xt}.t}{(e^t+1)^{\alpha}}=\sum_{n=0}^{\infty}\frac{d}{dx}E_n^{\alpha}(x)\frac{t^n}{n!}$$

or $$\sum_{n=0}^{\infty}E_n^{\alpha}(x)\frac{t^{n+1}}{n!}=\sum_{n=0}^{\infty}\frac{d}{dx}E_n^{\alpha}(x)\frac{t^n}{n!}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we get (6).

Particular case : For α=1, we get the property of ordinary Euler polynomials as

$$\frac{d}{dx}\left[E_n(x)\right]=E_n^{'}(x)=n.E_{n-1}(x) \qquad .....(7)$$

Property 2 (Addition formula) : For $n\geq 1$, we have

$$E_n^{\alpha}(x+1)+E_n^{\alpha}(x)=2E_n^{(\alpha-1)}(x) \qquad .....(8)$$

Proof : Replacing $x\to x+1$ in (1), we get

$$\frac{(2)^{\alpha}e^{(x+1)t}}{(e^t+1)^{\alpha}}=\sum_{n=0}^{\infty}E_n^{\alpha}(x+1)\frac{t^n}{n!} \qquad ....(9)$$

Adding equations (1) and (9), we get

$$\frac{(2)^{\alpha}e^{(x+1)t}}{(e^t+1)^{\alpha}}+\frac{(2)^{\alpha}e^{xt}}{(e^t+1)^{\alpha}}=\sum_{n=0}^{\infty}\left[E_n^{\alpha}(x+1)+E_n^{\alpha}(x)\right]\frac{t^n}{n!}$$

or $$\frac{(2)^{\alpha-1+1}e^{xt}\left[e^t+1\right]}{(e^t+1)^{\alpha-1+1}}=\sum_{n=0}^{\infty}\left[E_n^{\alpha}(x+1)+E_n^{\alpha}(x)\right]\frac{t^n}{n!}$$

or $$2\sum_{n=0}^{\infty}\left[E_n^{\alpha-1}(x)\right]\frac{t^n}{n!}=\sum_{n=0}^{\infty}\left[E_n^{\alpha}(x+1)+E_n^{\alpha}(x)\right]\frac{t^n}{n!}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we obtain the addition formula (8).

Particular case : When α=1, and using $E_n^0(x) = x^n$, we get the property of Euler polynomials[5] :

$$E_n(x+1) + E_n(x) = 2x^n \qquad ....(10)$$

Property 3. $E_n^\alpha(\alpha - x) = (-1)^n E_n^\alpha(x)$ ....(11)

Proof : Replacing $x \to \alpha - x$ in (1), we get

$$\frac{(2)^\alpha e^{(\alpha-x)t}}{(e^t+1)^\alpha} = \sum_{n=0}^{\infty} E_n^\alpha(\alpha - x)\frac{t^n}{n!}$$

or
$$\frac{(2)^\alpha e^{-xt}}{(1+e^{-t})^\alpha} = \sum_{n=0}^{\infty} E_n^\alpha(\alpha - x)\frac{t^n}{n!}$$

or
$$\sum_{n=0}^{\infty} E_n^\alpha(x)(-1)^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^\alpha(\alpha - x)\frac{t^n}{n!}$$

Comparing the coefficient of $\frac{t^n}{n!}$, we get (11).

Particular case . When α=1,we get property of the ordinary Bernoulli polynomials

$$E_n^\alpha(1-x) = (-1)^n E_n(x) \qquad .....(12)$$

Property 4. (Addition theorem) :

$$E_n^{\alpha+\beta}(x+y) = \sum_{k=0}^{n} E_k^\alpha(x) E_{n-k}^\beta(y) \qquad ....(13)$$

Proof : Putting $\alpha \to \alpha + \beta$ and x by $x + y$ in (1), we get

$$\frac{(2)^{(\alpha+\beta)} e^{(x+y)t}}{(e^t+1)^{(\alpha+\beta)}} = \sum_{n=0}^{\infty} E_n^{(\alpha+\beta)}(x+y)\frac{t^n}{n!}$$

or
$$\frac{(2)^{(\alpha)} e^{xt}}{(e^t+1)^\alpha}\frac{(2)^{(\beta)} e^{yt}}{(e^t+1)^\beta} = \sum_{n=0}^{\infty} E_n^{(\alpha+\beta)}(x+y)\frac{t^n}{n!}$$

or $$\sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n}{n!}.\sum_{n=0}^{\infty} E_n^{\beta}(y)\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{(\alpha+\beta)}(x+y)\frac{t^n}{n!}$$

By the Cauchy product rule for multiplying power series, we easily arrive at (13).

Property 5 : $$E_n^{\alpha}(x+y) = \sum_{k=0}^{n}\binom{n}{k}.E_k^{\alpha}(x).y^{n-k} \quad ....(14)$$

Proof : Replacing x by x+ y in generating function (1), we get

$$\frac{(2)^{\alpha} e^{(x+y)t}}{(e^t+1)^{\alpha}} = \sum_{n=0}^{\infty} E_n^{\alpha}(x+y)\frac{t^n}{n!}$$

or $$\frac{(2)^{\alpha} e^{xt} e^{yt}}{(e^t+1)^{\alpha}} = \sum_{n=0}^{\infty} E_n^{\alpha}(x+y)\frac{t^n}{n!}$$

or $$\sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n}{n!}.\sum_{n=0}^{\infty} y^n\frac{t^n}{n!} = \sum_{n=0}^{\infty} E_n^{\alpha}(x+y)\frac{t^n}{n!}$$

By the Cauchy's product rule for multiplying power series, we get (14).

Now if we put y=1 in (14) we get

$$E_n^{\alpha}(x+1) = \sum_{k=0}^{n}\binom{n}{k} E_k^{\alpha}(x). \quad ....(15)$$

Particular case : When $\alpha = 1$, we get the recurrence relation in terms of Euler polynomials :

$$E_n(x+1) = \sum_{k=0}^{n}\binom{n}{k} E_k(x). \quad ....(16)$$

Property 6 : $$E_n^{\alpha+1}(x) = \frac{2}{\alpha}E_{n+1}^{\alpha}(x) + \frac{2}{\alpha}(\alpha - x)E_n^{\alpha}(x) \quad ....(17)$$

Proof . Differentiating (1) w.r.t. 't' and multiplying both the sides by t, we get

$$\sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n.n}{n!} = \frac{(2)^{\alpha}.e^{xt}.x.t}{(e^t+1)^{\alpha}} - \frac{t.\alpha(2)^{\alpha+1}.e^{(x+1)t}}{2(e^t+1)^{\alpha+1}}$$

or $$\sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^n.n}{n!} = x\sum_{n=0}^{\infty} E_n^{\alpha}(x)\frac{t^{n+1}.}{n!} - \frac{\alpha}{2}\sum_{n=0}^{\infty} E_n^{\alpha+1}(x+1)\frac{t^{n+1}}{n!}$$

Equating the coefficients of $\frac{t^{n+1}}{n!}$, we find that

$$E_{n+1}^{\alpha}(x) = x.E_n^{\alpha}(x) - \frac{\alpha}{2}E_n^{\alpha+1}(x+1) \qquad ....(18)$$

But we know that by

$$E_n^{\alpha+1}(x) = 2.E_n^{\alpha}(x) - E_n^{\alpha+1}(x) \qquad ....(19)$$

Putting the value of (18) in (19), we obtain

$$E_{n+1}^{\alpha}(x) = xE_n^{\alpha}(x) - \frac{\alpha}{2}[2E_n^{\alpha}(x) - E_n^{\alpha+1}(x)]$$

or $$E_n^{\alpha+1}(x) = \frac{2}{\alpha}E_{n+1}^{\alpha}(x) + \frac{2}{\alpha}(\alpha - x)E_n^{\alpha}(x)$$

Particular case : (i) Putting x=0 in (17), we have

$$C_n^{\alpha+1} = \frac{2}{\alpha}C_{n+1}^{\alpha} + 2C_n^{\alpha} \qquad .....(20)$$

(ii) Also putting α=1 in (17), we get a known result for ordinary Euler polynomials [1] :

$$E_n^2(x) = 2.E_{n+1}(x) + 2(1-x)E_n(x) \qquad .....(21)$$

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