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PARTIAL DIFFERENTIAL EQUATIONS ASSOCIATED WITH SOME GENERALIZED FUNCTIONS OF TWO VARIABLES

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Abstract : In the present note we investigate a pair of partial differential equations of H-function of two variables for suitably constrained values of parameters. The partial differential equations for corresponding G-function of two variables and the generalized Kampé de Fériet function are obtained as particular cases and that of Appell's functions and H-function of one variable are found to be in agreement with those available in literature.

Keywords: Appell's functions, generalized Kampé de fériet function, G-function of two variables, H-function of two variables, Fox H-function.

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1. Introduction and Definitions

The role of differential equations is very important in the study of special functions. The differential equation for generalized hyper-geometric function in one variable is well known in the literature [4]. Further the differential equation of Meijer's *G*-function, is also available in the book by Luke [4]. However, the differential equation for general Fox *H*-function does not exist in literature, it exists only for *H*-function with rational parameters given by Pathak [6] (see also [3, p.

346]). As regards the hypergeometric functions of two variables, we find differential equations of Appell's functions and their confluent forms in the literature [2].

The H-function of two variables ([5], see also [9, p. 82]) is given by

$$H[x,y]=H^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\left|\begin{matrix} (a_j;\alpha_j,A_j)_{1,p_1} : (c_j,\gamma_j)_{1,p_2} ; (e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1} : (d_j,\delta_j)_{1,q_2} ; (f_j,F_j)_{1,q_3} \end{matrix}\right.\right]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta, \qquad ....(1)$$

where $$\phi_1(\xi,\eta)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)}, \qquad ....(2)$$

$$\theta_2(\xi)=\frac{\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)}, \qquad ....(3)$$

$$\theta_3(\eta)=\frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)}, \qquad ....(4)$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non-negative integers such that $p_i \geq n_i \geq 0,\ q_i \geq 0, q_i \geq m_j \geq 0$ $(i=1,2,3; j=2,3)$. All the A_j, α_j, B_j, β_j, γ_j, δ_j, E_j and F_j are assumed to be positive. The conditions of convergence of integral (1) and nature of contours L_1 and L_2 can be seen in the book by Srivastava et al. [9, p. 83].

Particularly for $\alpha_j = A_j = 1(j=1,\ldots,p_1)$, $\beta_j = B_j = 1(j=1,\ldots,q_1)$, $\gamma_j = 1(j=1,\ldots,p_2)$, $\delta_j = 1(j=1,\ldots,q_2)$, $E_j = 1(j=1,\ldots,p_3)$,

$F_j = 1(j = 1, \ldots, q_3)$, the *H*-function of two variables reduces to *G*-function of two variables $G[x, y]$ ([1], [7], see also [9, p. 88]).

For $n_1 = p_1, n_2 = p_2, n_3 = p_3, m_2 = m_3 = 1$, $q_2, q_3, x, y, a_j, b_j, c_j, d_j, e_j, f_j$ replaced by $q_2 + 1$, $q_3 + 1$, $-x$, $-y, 1 - a_j$, $1 - b_j, 1 - c_j, 1 - d_j, 1 - e_j, 1 - f_j$ respectively, $d_1 = \delta_1 = f_1 = F_1 = 1$, the *H*-function of two variables reduces to generalized Kampé de Fériet function $S[x, y]$ ([8], see also [9, p. 88]).

For $n_1 = p_1 = q_1 = 0$, the *H*-function of two variables breaks up into product of two *H*-functions of one variable [9, p. 90].

2. Main Results

The partial differential equations of *H*-function of two variables defined by(1) for suitably constrained values of some of its parameters are obtained in the following theorem and the partial differential equations of some functions that arise as its particular cases are obtained in the form of corollaries.

Theorem. *Let* $\theta \equiv x \partial/\partial x, \phi \equiv y \partial/\partial y$ *then the H-function of two variables defined by (1) satisfies the following pair of partial differential equations (5) and (6).*

$$\Bigg[(-1)^{\sum_{j=1}^{n_1}\alpha_j + \sum_{j=1}^{n_2}\gamma_j + \sum_{j=1}^{m_2}\delta_j}\, x \prod_{j=1}^{p_1}\prod_{i=1}^{\alpha_j}\left(a_j - \alpha_j\theta - A_j\phi - i\right)\prod_{j=1}^{p_2}\prod_{i=1}^{\gamma_j}\left(c_j - \gamma_j\theta - i\right)$$

$$-\prod_{j=1}^{q_1}\prod_{i=1}^{\beta_j}\left(1 - b_j + \beta_j\theta + B_j\phi - i\right)\prod_{j=1}^{q_2}\prod_{i=1}^{\delta_j}\left(1 - d_j + \delta_j\theta - i\right)\Bigg] f(x, y) = 0, \quad \ldots.(5)$$

where all α_j, β_j, γ_j and δ_j *are assumed to be positive integers and*

$$\Bigg[(-1)^{\sum_{j=1}^{n_1}A_j + \sum_{j=1}^{n_3}E_j + \sum_{j=1}^{m_3}F_j}\, y \prod_{j=1}^{p_1}\prod_{i=1}^{A_j}\left(a_j - \alpha_j\theta - A_j\phi - i\right)\prod_{j=1}^{p_3}\prod_{i=1}^{E_j}\left(e_j - E_j\phi - i\right)$$

$$-\prod_{j=1}^{q_1}\prod_{i=1}^{B_j}\left(1 - b_j + \beta_j\theta + B_j\phi - i\right)\prod_{j=1}^{q_3}\prod_{i=1}^{F_j}\left(1 - f_j + F_j\phi - i\right)\Bigg] f(x, y) = 0, \quad \ldots(6)$$

where all A_j, B_j, E_j *and* F_j *are assumed to be positive integers.*

Proof. Taking $f(x,y) = H[x,y]$ in (5), using definition (1), assuming all α_j, β_j, γ_j and δ_j to be positive integers, operating the differential operators θ and ϕ, using the results $\theta(x^m) = mx^m, \phi(y^n) = ny^n$ and doing some simplifications we get

$$\left[(-1)^{\sum_{j=1}^{n_1}\alpha_j+\sum_{j=1}^{n_2}\gamma_j+\sum_{j=1}^{m_2}\delta_j} x\prod_{j=1}^{p_1}\prod_{i=1}^{\alpha_j}\left(a_j-\alpha_j\theta-A_j\phi-i\right)\prod_{j=1}^{p_2}\prod_{i=1}^{\gamma_j}\left(c_j-\gamma_j\theta-i\right)\right.$$

$$\left.-\prod_{j=1}^{q_1}\prod_{i=1}^{\beta_j}\left(1-b_j+\beta_j\theta+B_j\phi-i\right)\prod_{j=1}^{q_2}\prod_{i=1}^{\delta_j}\left(1-d_j+\delta_j\theta-i\right)\right]H[x,y]$$

$$=-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{n_2}\Gamma\left(1-c_j+\gamma_j(\xi+1)\right)\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j\xi\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j(\xi+1)\right)\prod_{j=m_2+1}^{q_2}\Gamma\left(1-d_j+\delta_j\xi\right)}\frac{\prod_{j=1}^{n_3}\Gamma\left(1-e_j+E_j\eta\right)\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\Gamma\left(1-f_j+F_j\eta\right)}$$

$$\times\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j(\xi+1)+A_j\eta\right)x^{\xi+1}y^{\eta}d\xi d\eta}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j(\xi+1)-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j\xi+B_j\eta\right)}$$

$$-\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\frac{\prod_{j=1}^{n_2}\Gamma\left(1-c_j+\gamma_j\xi\right)\prod_{j=1}^{m_2}\Gamma\left(d_j-\delta_j(\xi-1)\right)}{\prod_{j=n_2+1}^{p_2}\Gamma\left(c_j-\gamma_j\xi\right)\prod_{j=m_2+1}^{q_2}\Gamma\left(1-d_j+\delta_j(\xi-1)\right)}\frac{\prod_{j=1}^{n_3}\Gamma\left(1-e_j+E_j\eta\right)\prod_{j=1}^{m_3}\Gamma\left(f_j-F_j\eta\right)}{\prod_{j=n_3+1}^{p_3}\Gamma\left(e_j-E_j\eta\right)\prod_{j=m_3+1}^{q_3}\Gamma\left(1-f_j+F_j\eta\right)}$$

$$\times\frac{\prod_{j=1}^{n_1}\Gamma\left(1-a_j+\alpha_j\xi+A_j\eta\right)x^{\xi}y^{\eta}d\xi d\eta}{\prod_{j=n_1+1}^{p_1}\Gamma\left(a_j-\alpha_j\xi-A_j\eta\right)\prod_{j=1}^{q_1}\Gamma\left(1-b_j+\beta_j(\xi-1)+B_j\eta\right)}=0$$

Proceeding as above, the partial differential equation (6) can also be verified.

Corollary 1. On specializing the H-function of two variables in the Theorem, to the G-function of two variables $G[x,y]$ [9, p. 88], we find that $f(x,y) = G[x,y]$ satisfies the following pair of partial differential equations

$$\left[(-1)^{n_1+n_2+m_2}\, x\prod_{j=1}^{p_1}\left(a_j-\theta-\phi-i\right)\prod_{j=1}^{p_2}\left(c_j-\theta-i\right)\right.$$

$$\left.-\prod_{j=1}^{q_1}\left(1-b_j+\theta+\phi-i\right)\prod_{j=1}^{q_2}\left(1-d_j+\theta-i\right)\right]f(x,y)=0 \quad ....(7)$$

and

$$\left[(-1)^{n_1+n_3+m_3}\, y\prod_{j=1}^{p_1}\left(a_j-\theta-\phi-i\right)\prod_{j=1}^{p_3}\left(e_j-\phi-i\right)\right.$$

$$\left.-\prod_{j=1}^{q_1}\left(1-b_j+\theta+\phi-i\right)\prod_{j=1}^{q_3}\left(1-f_j+\phi-i\right)\right]f(x,y)=0. \quad .....(8)$$

Corollary 2. On specializing the H-function of two variables in the Theorem, to generalized Kampé de Fériet function of two variables $S[x,y]$ [9, p. 88 Eq. (6.4.2)], we get the following pair of partial differential equations for $f(x,y)=S[x,y]$.

$$\left[\prod_{j=1}^{p_1}\prod_{i=1}^{\alpha_j}\left(a_j+\alpha_j\theta+A_j\phi+i-1\right)\prod_{j=1}^{p_2}\prod_{i=1}^{\gamma_j}\left(c_j+\gamma_j\theta+i-1\right)\right.$$

$$\left.-x^{-1}\theta\prod_{j=1}^{q_1}\prod_{i=1}^{\beta_j}\left(b_j+\beta_j\theta+B_j\phi-i\right)\prod_{j=1}^{q_2}\prod_{i=1}^{\delta_j}\left(d_j+\delta_j\theta-i\right)\right]f(x,y)=0 \quad .....(9)$$

where all α_j, β_j, γ_j and δ_j are assumed to be positive integers.

$$\left[\prod_{j=1}^{p_1}\prod_{i=1}^{A_j}\left(a_j+\alpha_j\theta+A_j\phi+i-1\right)\prod_{j=1}^{p_3}\prod_{i=1}^{E_j}\left(e_j+E_j\phi+i-1\right)\right.$$

$$\left.-y^{-1}\phi\prod_{j=1}^{q_1}\prod_{i=1}^{B_j}\left(b_j+\beta_j\theta+B_j\phi-i\right)\prod_{j=1}^{q_3}\prod_{i=1}^{F_j}\left(f_j+F_j\phi-i\right)\right]f(x,y)=0 \, ...(10)$$

where all A_j, B_j, E_j and F_j are assumed to be positive integers.

On specializing H-function of two variables to Appell's functions as given in the book by Srivastava, Gupta and Goyal [9, p. 89], we obtain the partial differential equations of Appell's functions which match with the partial differential equations given in the book by Erdélyi, Magnus, Oberhettinger and Tricomi [2, pp. 233-234].

On reducing the H-function of two variables to product of two H-functions of one variable as given in the book by Srivastava, Gupta and Goyal [9, p. 90], we obtain differential equation of H-function of one variable which matches with the differential equation of H-function given in the book by Kiryakova [3, p. 346] for the case $r = 1$ therein.

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References

1. Agrawal, R.P. (1965). An extension of Meijer's G-function, *Proc. Nat. Inst. Sci. India Part A* **31**, 536-546.
2. Erdélyi, A., Magnus, W., Oberhettinger F., and Tricomi, F.G. (1953), *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York-Toronto-London.
3. Kiryakova, V. (1994), *Generalized Fractional Calculus and Applications*, Pitman Research Notes in Mathematics series No. 301, Longman Scientific and Technical, Harlow, Essex, 1994.
4. Luke, Y.L. (1969). *The Special Functions and Their Approximations*, Vol. I, Academic Press, New York and London.
5. Mittal, P.K., and Gupta, K.C. (1972). An integral involving generalized function of two variables, *Proc. Indian Acad. Sci. Sect. A* **75**, 117-123.
6. Pathak, R.S. (1972). A general differential equation satisfied by a special function, *Progress of Math.* **6(1-2)**, 46-50.
7. Sharma, B.L. (1965). On a generalized function of two variables, *I. Ann. Soc. Sci. Bruxeles Ser.* **179**, 26-40.
8. Srivastava, H.M., and Daoust, Martha C. (1969). Certain generalized Neumann expansions associated with the Kampe de Feriet function, *Nederl. Akad. Wetensch. Proc. Ser. A 72 Indag. Math.* **31**, 449-457.
9. Srivastava, H.M., Gupta, K.C., and Goyal, S.P. (1981). *The H-function of One and Two Variables with Applications*, South Asian Publishers, New Delhi.