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EXACT AND APPROXIMATE TESTS OF A CONDITIONALLY SPECIFIED TEST PROCEDURE IN FOUR-STAGE UNBALANCED NESTED DESIGNS

ANITA MEHTA AND M. S. DULAWAT

Department of Mathematics & Statistics, Mohanlal Sukhadia University, Udaipur-313001(INDIA)
E-mail : anita19dsm@rediffmail.com

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Abstract : In this paper, a study of exact and approximate tests for four-stage unbalanced nested designs considering random effects model at the third and the fourth-stage has been made. It seems that the expressions derived for these tests would be statistically advantageous.

Keywords: Exact test, approximate test, nested designs, MSS (mean sum of squares), Chi-square test and F-test.

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1. Introduction

The balanced data are those which have equal number of observations in all the subclasses of the model. For other properties of balanced data refer Searle [6]. On the contrary sides the unbalanced data are those wherein the number of observations in the subclasses of the model is not all the same.

In nested designs or hierarchical classification the entire experimental area under study is subdivided into a large number of relatively smaller areas which are the first-stage classes.. Further, each of these areas or classes is subdivided into various sub areas or subclasses which are second-stage classes. This process continuous till we get the four-stage classes. In unbalanced nested designs, the number of observations or units or the number of division of a particular class at a particular stage are not all equal.

Exact and approximate tests using unbalanced nested design have been discussed by Tietjen [7] and Tietjen and Moore [8] respectively. Bozivich, Bancroft and Hartely [2] and Brar [3].

derived exact and approximate tests for three-stage unbalanced design. In this paper, we have developed the process of testing the hypothesis $H_0 : \sigma_a^2 = 0$ using conventional F-test. Here we consider the linear model for four-stage unbalanced design as

$$Y_{ijkl} = \mu + a_i + b_{ij} + c_{ijk} + e_{ijkl} \quad ...(1)$$

All terms in right hand side of the model except μ are assumed to be independent and normally distributed random variables with zero means and variances $\sigma_a^2, \sigma_b^2, \sigma_c^2$ and σ_e^2 respectively.

Table1. Analysis of variance table for four-stage unbalanced nested designs

Source of variation	Degrees of freedom	Mean squares	E(MSS)
Between A classes	$\eta_4 = a - 1$	$Y'Q_4Y = M_4$	$\sigma_e^2 + \tau_3\sigma_c^2 + \tau_4\sigma_b^2 + \tau_5\sigma_a^2$
Between B classes within A classes	$\eta_3 = \sum_i^a b_i - a$	$Y'Q_3Y = M_3$	$\sigma_e^2 + \tau_1\sigma_c^2 + \tau_2\sigma_b^2$
Between C classes within B classes	$\eta_2 = \sum_i^a \sum_j^{b_i} c_{ij} - \sum_i^a b_i$	$Y'Q_2Y = M_2$	$\sigma_e^2 + \tau_0\sigma_c^2$
within C classes	$\eta_1 = N - \sum_i^a \sum_j^{b_i} c_{ij}$	$Y'Q_1Y = M_1$	σ_e^2

where $Q_1 = \left[I - \sum_i^a \sum_j^{b_i} \sum_k^{c_{ij}} \frac{J_{n_{ijk}}}{n_{ijk}} \right], Q_2 = \left[\sum_i^a \sum_j^{b_i} \sum_k^{c_{ij}} \frac{J_{n_{ijk}}}{n_{ijk}} - \sum_i^a \sum_j^{b_i} \frac{J_{n_{ij}}}{n_{ij}} \right]$

$$Q_3 = \left[\sum_i^a \sum_j^{b_i} \frac{J_{n_{ij0}}}{n_{ij0}} - \sum_i^a \frac{J_{n_i}}{n_i}\right], \quad Q_4 = \left[\sum_i^a \frac{J_{n_i}}{n_i} - \frac{J_n}{n}\right],$$

$$\tau_0 = \frac{(N-k_6)}{(c-b)}, \qquad \tau_1 = \frac{(k_6-k_5)}{(b-a)}, \tau_2 = \frac{(N-k_4)}{(b-a)},$$

$$\tau_3 = \frac{(k_5-k_3)}{(a-1)}, \tau_4 = \frac{(k_4-k_2)}{(a-1)}, \tau_5 = \frac{(N-k_1)}{(a-1)}.$$

where J_n denote an $(n \times n)$ matrix of unit elements and $\sum_i^k A_i$, the direct sum of matrices $A_1, A_2, \ldots\ldots, A_k$ as in Searle (1971) and

$$b = \sum_i^a b_i, \; c = \sum_i^a \sum_j^{b_i} c_{ij}, \; N = \sum_i^a \sum_j^{b_i} \sum_k^{c_{ij}} n_{ijk}, \; n_{i00} = \sum_j \sum_k n_{ijk}, \; n_{ij0} = \sum_k n_{ijk}.$$

The k's that define in the above relations are functions of n_{ijk} 's namely,

$$k_1 = \frac{\sum_i^a n_{i00}^2}{N}, \; k_2 = \frac{\sum_i^a \sum_j^{b_i} n_{ij0}^2}{N}, \; k_3 = \frac{\sum_i^a \sum_j^{b_i} \sum_k^{c_{ij}} n_{ijk}^2}{N},$$

$$k_4 = \sum_i^a \left[\sum_j^{b_i} \frac{n_{ij0}^2}{n_{i00}}\right], \; k_5 = \sum_i^a \left[\sum_j^{b_i} \sum_k^{c_{ij}} \frac{n_{ijk}^2}{n_{i00}}\right], \; k_6 = \sum_i^a \sum_j^{b_i} \left[\sum_k^{c_{ij}} \frac{n_{ijk}^2}{n_{ij0}}\right].$$

where Y_{ijkl} is equal to the sum of general mean μ and independent normal variables a_i, b_{ij}, c_{ijk} and e_{ijkl}, the vector $\underline{Y}$ is distributed as $N(\mu, \underline{J}, M)$, where $\underline{J}$ denotes an all one column vector. The variance-covariance matrix M can be expressed as

$$M = \sigma_e^2 I + \sigma_c^2 \sum\sum\sum J_{n_{ijk}} + \sigma_b^2 \sum\sum J_{n_{ij}} + \sigma_a^2 \sum J_{n_i} \qquad \text{....(2)}$$

In testing the hypothesis $H_0 : \sigma_a^2 = 0$ against the alternative hypothesis $H_1 : \sigma_a^2 > 0$. We have seen from ANOVA table that an exact test of the hypothesis $H_0 : \sigma_a^2 = 0$ is not possible in general because $\tau_1 \neq \tau_3$ and $\tau_4 \neq \tau_2$. When $n_{ijk} = n$ and $c_{ij} = k$ for all i, j and k, an exact test is available because under this condition $\tau_1 = \tau_3$ and $\tau_4 = \tau_2$.

Approximate Tests

When there is an unbalance at the third and the fourth-stage, we construct approximate F-test for testing the variance component of treatment effects using Satterthwaite's [5] procedures. These tests are applicable where the design structure is such that the mean squares of main classes and subclasses are distributed independently and have chi-square type distribution which generally does not hold for unbalanced designs.

Approximate F- tests with positive coefficients

If M' is a linear combination of independent mean square with expectation equal to $\left(\sigma_e^2 + \tau_3\sigma_c^2 + \tau_4\,\sigma_b^2\right)$ i.e. $E(M_4)/H_0$, then test statistic for testing H_0 will be

$$F_1 = \frac{M_4}{M'} \qquad ...(3)$$

where $M_4 = \sigma_e^2 + \tau_3\sigma_c^2 + \tau_4\sigma_b^2$ under $H_0 : \sigma_a^2 = 0$ and $M' = \hat{\sigma}_e^2 + \tau_3\,\hat{\sigma}_c^2 + \tau_4\,\hat{\sigma}_b^2$

To obtain estimates of the variance components is accomplished by equating expected and observed mean squares in the ANOVA Table 1 as follows:

$$\hat{\sigma}_e^2 = M_1 \quad \hat{\sigma}_c^2 = \frac{(M_2 - M_1)}{\tau_0} \text{ and } \hat{\sigma}_b^2 = \frac{\left[M_3 - M_1 - \frac{\tau_1}{\tau_0}(M_2 - M_1)\right]}{\tau_2}$$

$$M' = \left[C_3M_3 + (C_2 - C_1C_3)M_2 + (1 - C_2 - C_3 + C_1C_3)M_1\right] \qquad ...(4)$$

On putting $\frac{\tau_1}{\tau_0}=C_1$, $\frac{\tau_3}{\tau_0}=C_2$, $\frac{\tau_4}{\tau_2}=C_3$

where $(1-C_2-C_3+C_1C_3)<1$, $(C_2-C_1C_3)>0$, $C_3<1$.

Distribution of F_1

For positive coefficients $F_1=\frac{M_4}{M'}$ M_1 is distributed as $\sigma_1^2\frac{\chi^2_{\eta_1}}{\eta_1}$ and M_2 and M_3 are distributed as $d_2\frac{\chi^2_{\upsilon_2}}{\eta_2}$ and $d_3\frac{\chi^2_{\upsilon_3}}{\eta_3}$, respectively (Box [1]). Then M' is distributed as

$$C_3d_3\frac{\chi^2_{\upsilon_3}}{\eta_3}+(C_2-C_1C_3)d_2\frac{\chi^2_{\upsilon_2}}{\eta_2}+(1-C_2-C_3-C_1C_3)\sigma_1^2\frac{\chi^2_{\eta_1}}{\eta_1} \qquad ...(5)$$

By Satterthwaite's approximation. M' has approximately a chi-square type distribution i.e. M' is approximately distributed as $g_1\chi^2_{h_1}$ from (5) we have

$E(M')=E(g_1\chi^2_{h_1})=g_1h_1=\sigma_M^2$ (say) and $V(M')=Var(g_1\chi^2_{h_1})=2g_1^2h_1$, where

$$g_1=\frac{\left[C_3^2\frac{\sigma_3^4}{\upsilon_3}+(C_2-C_1C_3)^2\frac{\sigma_2^4}{\upsilon_2}+(1-C_2-C_3+C_1C_3)^2\frac{\sigma_1^4}{\eta_1}\right]}{\left[C_3\sigma_3^2+(C_2-C_1C_3)\sigma_2^2+(1-C_2-C_3+C_1C_3)\sigma_1^2\right]} \text{ and}$$

$$h_1=\frac{\left[C_3\sigma_3^2+(C_2-C_1C_3)\sigma_2^2+(1-C_2-C_3+C_1C_3)\sigma_1^2\right]^2}{\left[C_3^2\frac{\sigma_3^4}{\upsilon_3}+(C_2-C_1C_3)^2\frac{\sigma_2^4}{\upsilon_2}+(1-C_2-C_3+C_1C_3)^2\frac{\sigma_1^4}{\eta_1}\right]}$$

where $\sigma_2^2=\frac{d_2\upsilon_2}{\eta_2}=E(M_2)$ and $\sigma_3^2=\frac{d_2\upsilon_3}{\eta_3}=E(M_3)$ and the M.S.S. M_4 is approximately distributed as $d_4\chi^2_{\upsilon_4}/\eta_4$ Therefore, F_1 is approximately distributed under the null hypothesis H_0 as $F_1\sim\frac{\sigma_4^2}{\sigma_M^2}.F(\upsilon_4,h_1)$

where $\sigma_4^2=\frac{d_4\upsilon_4}{\eta_4}=E(M_4)$ and $(1-C_2-C_3+C_1C_3)>0$, $(C_2-C_1C_3)>0$, $C_3>0$.

Approximate F-tests with mixed coefficients

According to Gaylor and Hooper [4] the linear combination of mean squares MS (say) is grouped in two parts i.e. MS = $MS_1 - MS_2$, where MS_1 is a linear combination of mean squares with positive coefficients having m_1 degrees of freedom and MS_2, the linear combination of mean squares with negative coefficients having m_2 degrees of freedom. The simulation studies made by them suggest that $\hat{f}_0 \frac{MS}{E(MS)}$ is approximately distributed as $\chi^2_{\hat{f}_0}$ provided

$$\hat{\rho} = \frac{MS_1}{MS_2} \geq F_{m_2,m_1}(0.975) \cdot F_{m_1,m_2}(0.50) \text{ for } m_1 \leq 100 \text{ and } m_1 \leq 2m_2$$

where $\hat{f}_0 = \frac{(\hat{\rho}-1)^2}{\left(\frac{\hat{\rho}^2}{m_1} + \frac{1}{m_2}\right)}$ and $\hat{\rho} = \frac{MS_1}{MS_2} \geq 1$

Under the above condition, Satterthwaite's [5] procedure can be used on functions of mean squares that involve differences as well as sums (Searle [6]). Thus, the following test statistics are constructed using negative coefficients. In the statistic F_1, the coefficients $(1 + C_1 C_3 - C_2 - C_3)$ and $(C_2 - C_1 C_3)$ both are negative then

$$F_1' = \frac{M_4}{[C_3 M_3 - (C_1 C_3 - C_2) M_2 - (C_2 + C_3 - C_1 C_3 - 1) M_1]} \qquad ....(6)$$

where $(C_1 C_3 - C_2)$ and $(C_2 + C_3 - 1 - C_1 C_3)$ are absolute values of the coefficients $(C_2 - C_1 C_3)$ and $(1 - C_2 - C_3 + C_1 C_3)$. If one of these coefficients is negative then statistic will be in following forms

$$F_1'' = \frac{M_4}{[C_3 M_3 - (C_1 C_3 - C_2) M_2 + (1 + C_1 C_3 - C_2 - C_3) M_1]} \qquad ...(7)$$

$$F_1''' = \frac{M_4}{[C_3 M_3 + (C_2 - C_1 C_3) M_2 - (C_2 + C_3 - 1 - C_1 C_3) M_1]} \qquad ...(8)$$

For F_1', MS_2 i.e. $(C_1C_3 - C_2)M_2 + (C_2 + C_3 - 1 - C_1C_3)M_1$ is approximately distributed (by Satterthwaite's approximation) as $g_1'\chi^2_{h_1'}$ taking expectation and variance of MS_2 and solving, we obtain

$$g_1' = \frac{\left[(C_1C_3 - C_2)^2 \frac{\sigma_2^4}{\upsilon_2} + (C_2 + C_3 - C_1C_3 - 1)^2 \frac{\sigma_1^4}{\eta_1}\right]}{\left[(C_1C_3 - C_2)\sigma_2^2 + (C_2 + C_3 - C_1C_3 - 1)\sigma_1^2\right]} \quad \text{and}$$

$$h_1' = \frac{\left[(C_1C_3 - C_2)\sigma_2^2 + (C_2 + C_3 - C_1C_3 - 1)\sigma_1^2\right]^2}{\left[(C_1C_3 - C_2)^2 \frac{\sigma_2^4}{\upsilon_2} + (C_2 + C_3 - C_1C_3 - 1)^2 \frac{\sigma_1^4}{\eta_1}\right]}$$

Where $\sigma_2^2 = \frac{d_2\upsilon_2}{\eta_2} = E(M_2)$ under H_0. Therefore MS_2 is approximately distributed as $g_1^I h_1^I \frac{\chi^2_{h_1^I}}{h_1^I}$. For F_1' to be valid, the following condition must be satisfied:

$$\frac{C_3M_3}{\left[(C_1\,C_3 - C_2)M_2 + (C_2 + C_3 - 1 - C_1C_3)M_1\right]} \geq F_{h_1',\upsilon_3}(0.975)\cdot F_{\upsilon_3,h_1'}(0.50) \qquad ...(9)$$

For $\upsilon_3 \leq 100$ *and* $\upsilon_3 \leq 2h_1'$. Consider the statistic F_1'' given by (7) where $MS_1 = C_3\,M_3 + (1 + C_1\,C_3 - C_2 - C_3)M_1$ and $MS_2 = (C_1\,C_3 - C_2)M_2$. The linear combination of MS_1 (by Satterthwaite's approximation) is approximately distributed as $g_1''\chi^2_{h_1''}$. Taking expectation and variance of MS_1 and solving we have

$$g_1'' = \frac{\left[\frac{C_3^2\sigma_3^4}{\upsilon_3} + (1 + C_1C_3 - C_2 - C_3)^2 \frac{\sigma_1^4}{\eta_1}\right]}{\left[C_3\sigma_2^3 + (1 + C_1C_3 - C_2 - C_3)\sigma_1^2\right]}$$

and $$h_1'' = \frac{\left[C_3\sigma_2^3 + (1 + C_1C_3 - C_2 - C_3)\sigma_1^2\right]^2}{\left[\frac{C_3^2\sigma_3^4}{\upsilon_3} + (1 + C_1C_3 - C_2 - C_3)^2 \frac{\sigma_1^4}{\eta_1}\right]}$$

where $E(M_3)=\sigma_3^2=\frac{d_3\upsilon_3}{\eta_3}$ *under* H_0 Thus, MS_1 is approximately distributed as

$g_1''h_1''\frac{\chi^2_{h_1''}}{h_1''}$ and for F_1'' to be valid, the following condition must be satisfied.

$$\frac{\left[C_3\,M_3\;+\;(1\;+\;C_1\,C_3\;-C_2\;-C_3)\,M_1\right]}{(C_1C_3-C_2)M_2}\geq F_{\upsilon_2,h_1''}(0.975)\cdot F_{h_1'',\upsilon_2}(0.50)\quad\text{....(10)}$$

For $h_1''\leq 100$ and $h_1''\leq 2\upsilon_2$

Consider the statistic F_1''' given by (8). Where $MS_1=C_3M_3+(C_2-C_1C_3)M_2$ and $MS_2=(C_2+C_3-C_1C_3-1)\,M_1$ The linear combination of MS_1 is approximately distributed as $\left[\left(C_3d_3\upsilon_3\frac{\chi^2_{\upsilon_3}}{\upsilon_3}\right)+\left((C_2-C_1C_3)d_2\upsilon_2\frac{\chi^2_{\upsilon_2}}{\upsilon_2}\right)\right]$

where under $H_0,\sigma_3^2=\frac{d_3\upsilon_3}{\eta_3}=E(M_3)$ and $\sigma_2^2=\frac{d_2\upsilon_2}{\eta_2}=E(M_2)$.

By Satterthwaite's approximation MS_1 is approximately distributed as $g_1'''\chi^2_{h_1'''}$. Taking expectation and variance of MS_1 and solving, we obtain

$$g_1'''=\frac{\left[\frac{C_3^2\sigma_3^4}{\upsilon_3}+(C_2-C_1C_3)^2\frac{\sigma_2^4}{\upsilon_2}\right]}{\left[C_3\sigma_2^3+(C_2-C_1C_3)\sigma_2^2\right]}\quad\text{and}\quad h_1'''=\frac{\left[C_3\sigma_2^3+(C_2-C_1C_3)\sigma_2^2\right]^2}{\left[\frac{C_3^2\sigma_3^4}{\upsilon_3}+(C_2-C_1C_3)^2\frac{\sigma_2^4}{\upsilon_2}\right]}$$

Thus, $MS_1$is approximately distributed as $g_1'''\,h_1'''\frac{\chi^2_{h_1'''}}{h_1'''}$ and F_1''' to be valid, the following condition must be satisfied.

$$\frac{\left[C_3\,M_3+(C_2-C_1\,C_3\,)M_2\right]}{\left[(C_2+C_3-C_1\,C_3-1)M_1\right]}\geq F_{\eta_1,h_1'''}(0.975)\cdot F_{h_1''',\eta_1}(0.50)\qquad\text{...(11)}$$

for $h_1'''\leq 100$ and $h_1'''\leq 2\eta_1$.

Distribution of F_1'

The statistic F_1' can be written as

$$F_1' = \frac{M_4}{MSN} \qquad ....(12)$$

where $MSN = MS_1 - MS_2 = [C_3M_3 - (C_1 C_3 - C_2)M_2 - (C_2 + C_3 - C_1 C_3 - 1)M_1]$

and which is approximately distributed as $\left[C_3\sigma_3^2 \frac{\chi^2_{\upsilon_3}}{\upsilon_3} - g_1' h_1' \frac{\chi^2_{h_1'}}{h_1'}\right]$ where

$\sigma_3^2 = E(M_3)/H_0 = \frac{d_3\upsilon_3}{\eta_3}$ and $g_1'h_1' = E[(C_1 C_3 - C_2)M_2 + (C_2 + C_3 - C_1 Cl - 1)M_1]/H_0$.

Following Satterthwaite's [5] formula for linear combination of mean squares with negative coefficients, MSN is approximately distributed as $g_1^* \chi^2_{h_1^*}$. Taking expectation and variance of MSN and solving, we obtain

$$g_1^* = \frac{\left[\frac{C_3^2\sigma_3^4}{\upsilon_3} - \frac{(g_1' h_1')^2}{h_1'}\right]}{\left[C_3\sigma_3^2 - g_1' h_1'\right]} \quad \text{and} \quad h_1^* = \frac{\left[C_3\sigma_3^2 - g_1' h_1'\right]^2}{\left[\frac{C_3^2\sigma_3^4}{\upsilon_3} - \frac{(g_1' h_1')^2}{h_1'}\right]}$$

Therefore, $F_1' = \frac{M_4}{MSN}$ is approximately distributed as

$$\frac{\left(\frac{d_4\upsilon_4}{\eta_4}\right)\left(\frac{\chi^2_{\upsilon_4}}{\upsilon_4}\right)}{\left(g_1^* h_1^*\right)\left(\frac{\chi^2_{h_1^*}}{h_1^*}\right)} \text{ or } \frac{E(M_4)}{E(MSN)} F(\upsilon_4, h_1^*) \qquad ...(13)$$

and thus under H_0, F_1' is approximately distributed as $F_1' \sim F(\upsilon_4, h_1^*)$

Distribution of F_1''

The statistic F_1'' can be written as

$$F_1'' = \frac{M_4}{MSN} \quad ...(14)$$

where $MSN = MS_1 - MS_2 = [C_3M_3 - (C_1C_3 - C_2)M_2 + (C_1C_3 + 1 - C_2 - C_3)M_1]$ and which is approximately distributed as $\left[g_1''h_1''\frac{\chi^2_{h_1''}}{h_1''} - (C_1C_3 - C_2)\sigma_2^2\frac{\chi^2_{\upsilon_2}}{\upsilon_2}\right]$

where $\sigma_2^2 = E(M_2)/H_0 = \frac{d_2\upsilon_2}{\eta_2}$ and $g_1''h_1'' = E[C_3M_3 + (C_1C_3 + 1 - C_2 - C_3)M_1]/H_0$.

Following Satterthwaite's formula for linear combination of mean squares with negative coefficients, MSN is approximately distributed as $g_1^{**}\chi^2_{h_1^{**}}$.Taking expectation and variance of MSN and solving, we obtain

Therefore, $F_1'' = \frac{M_4}{MSN}$ is approximately distributed as

$$\frac{\left(\frac{d_4\upsilon_4}{\eta_4}\right)\left(\frac{\chi^2_{\upsilon_4}}{\upsilon_4}\right)}{\left(g_1^{**}h_1^{**}\right)\left(\frac{\chi^2_{h_1^{**}}}{h_1^{**}}\right)} \text{ or } \left[\frac{E(M_4)}{E(MSN)}\right] F(\upsilon_4, h_1^{**}) \quad ....(15)$$

and thus under H_0, F_1'' is approximately distributed as $F_1'' \sim F(\upsilon_4, h_1^{**})$

Distribution of F_1'''

The statistic F_1''' can be written as

$$F_1''' = \frac{M_4}{MSN} \quad ...(16)$$

where $MSN = MS_1 - MS_2 = [C_3M_3 + (C_2 - C_1C_3)M_2 - (C_2 + C_3 - C_1C_3 - 1)M_1]$ and which is approximately distributed as

$$\left(g_1'''h_1'''\frac{\chi^2_{h_1'''}}{h_1'''} - (C_2 + C_3 - C_1C_3 - 1)\sigma_1^2\frac{\chi^2_{\eta_1}}{\eta_1}\right)$$

where $\sigma_1^2 = E(M_1)$ and $g_1'''h_1''' = E(MS_1)/H_0$. Following Satterthwaite's formula for linear combination of mean squares with negative coefficients, MSN is approximately distributed as $g_1^{****}\chi^2_{h_1^{***}}$ .Taking expectation and variance of MSN and solving, we obtain

$$g_1^{****}=\frac{\left[\frac{(g_1'''h_1''')^2}{h_1'''}-\frac{(C_2+C_3-C_1C_3-1)^2\sigma_1^4}{\eta_1}\right]}{\left[(g_1'''h_1''')-(C_2+C_3-C_1C_3-1)\sigma_1^2\right]}$$

and $$h_1^{****}=\frac{\left[(g_1'''\,h_1''')-(C_2+C_3-C_1C_3-1)\sigma_1^2\right]^2}{\left[\frac{(g_1'''\,h_1''')^2}{h_1'''}-\frac{(C_2+C_3-C_1C_3-1)^2\sigma_1^4}{\eta_1}\right]}$$

Therefore, $F_1'''=\frac{M_4}{MSN}$ is approximately distributed as $\frac{\left(\frac{d_4\upsilon_4}{\eta_4}\right)\left(\frac{\chi^2_{\upsilon_4}}{\upsilon_4}\right)}{\left(g_1^{****}h_1^{****}\right)\left(\frac{\chi^2_{h_1^{***}}}{h_1^{****}}\right)}$ or as

$$\left[\frac{E(M_4)}{E(MSN)}\right]F\left(\upsilon_4, h_1^{****}\right) \qquad ....(17)$$

Thus, under H_0, F_1''' is approximately distributed as $F_1'''\sim F\left(\upsilon_4, h_1^{****}\right)$.

Conclusions:

(i). On considering both the coefficient of F as positive, we obtained M' which has approximately chi-square type distribution and the statistic F_1 is approximately distributed as central F distribution with respective degrees of freedom (υ_4, h_1).

(ii). On considering both the coefficient of F as negative we found that MSN is approximately distributed mjjkjkjlas $g_1^{*}\chi^2_{h_1^*}$ and F-statistic is distributed as $F_1'\sim F(\upsilon_4, h_1^*)$.

(iii). On considering the coefficient $(C_1C_3-C_2)$ as negative and $(1+C_1C_3-C_2-C_3)$ as positive we have seen that MSN is approximately distributed as $g_1^{**}\chi^2_{h_1^{**}}$ and F- statistic is distributed as $F_1''\sim F\left(\upsilon_4, h_1^{**}\right)$.

(iv). On considering the coefficient $(C_2 + C_3 - C_1 C_3 - 1)$ as negative and $(C_2 - C_1 C_3)$ as positive we get MSN is approximately distributed as $g_1^{***} \chi^2_{h_1^{***}}$ and F-statistic is distributed as $F_1''' \sim F(\upsilon_4, h_1^{***})$.

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